Compactness and normality in abstract logics

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Abstract

We generalize a theorem of Mundici relating compactness of a regular logic $L$ to a strong form of normality of the associated spaces of models. Moreover, it is shown that compactness is in fact equivalent to ordinary normality of the model spaces when $L$ has uniform reduction for infinite disjoint sums of structures. Some applications follow. For example, a countably generated logic is countably compact if and only if every clopen class in the model spaces is elementary. The model spaces of $L(Q_\omega)$ are not normal for vocabularies of uncountable power $\geq \omega_1$. It also follows that first-order logic is the only finite-dependence logic having normal model spaces and satisfying at the same time the downward Löwenheim–Skolem theorem and uniform reduction for pairs.

0. Introduction

Some facts in abstract model theory are purely topological, as we intend to illustrate in this paper. By exploiting the very basic properties of the spaces $E_\tau(L)$ of structures of type $\tau$, topologized by the $L$-elementary classes of a logic $L$, we study the connection between compactness and normality of these spaces. This topological approach has been utilized by Mundici [6] who shows several theorems relating compactness of a small regular logic to other properties of the associated spaces of models. We notice the following one, where $S_\tau(L)$ denotes the quotient of $E_\tau(L)$ by $L$-elementary equivalence [6, Theorem 2.3].
The following are equivalent for a small logic $L = L_{\omega_1}(Q^i : i \in I)$ closed under relativizations:

(i) $L$ is compact.

(ii) Any two closed sets in $S_c(L)$ may be separated by clopen sets of finite type (equivalently, by $L$-elementary classes of finite type).

Property (ii) may be seen as a strong form of normality of the spaces $S_c(L)$. In terms of semantical consequence it means: given sets of sentences $\{ \phi_i \}, \{ \psi_j \}$ in $L(\tau)$ such that $\bigwedge \phi_i \models \bigvee \psi_j$, then there is $\theta \in L(\tau)$ of finite type such that $\bigwedge \phi_i \models \theta \models \bigvee \psi_j$.

Mundici's proof of this theorem is based on the characterization of compactness via the existence of noncofinal elementary extensions of definable linear orders due to Väänänen [9, Theorem 1]; see also [5, Theorem 1.2.2]. In this note we give a purely topological proof in which model theory does not enter, except in a trivial way, yielding the theorem for more general logics. To this end, we prove first a general property of uniform spaces from which Mundici's characterization and a similar characterization for $\kappa$-compactness follow (Section 2). In addition, using a topological theorem of Noble [7], we show that if the logic has a weak form of uniform reduction for arbitrary disjoint sums, compactness is in fact equivalent to normality of the spaces of structures, and $\kappa$-compactness is equivalent, for uncountable $\kappa$, to normality with respect to closed classes defined by theories of power $\kappa$ (Section 4). These results hold for any small logic closed under finite Boolean operations and relativizations. We do not need the logic to be closed under substitutions, and the results apply to monadic logics.

Logics are defined as in [2, Definition 1.1.1], although we consider single-sorted logics only and we do not assume the domain of a logic to contain all possible vocabularies (see Remark 5). In addition, all logics are assumed to be small extensions of $L_{\omega_1 \omega}$ for the vocabularies in which they are defined (however, we utilize only tiny fragments of $L_{\omega_1 \omega}$), and to be closed under finite conjunctions, negations and relativizations to monadic atomic predicates. Any other closure condition will be explicitly stated. $A \models \tau$ and $A \models P^A$ will denote respectively the reduct of the structure $A$ to the vocabulary $\tau$ and the relativization of $A$ to the interpretation of the predicate $P$. For any unexplained concepts and notation we refer to [2].

We assume that the reader is acquainted with the basic facts about uniform spaces (cf. [10]). Recall that a uniformity may be given in terms of a family of pseudometrics, and the notions of Cauchy net, convergence of nets, complete space, totally bounded space, etc., may be expressed in terms of such a family.

1. Spaces of structures

Given a logic $L$ and a vocabulary $\tau$, let $E_\tau(L)$ be the large topological space of structures of type $\tau$, topologized by the $L$-elementary classes: $\text{Mod}(\phi)$, $\phi \in L(\tau)$,
as an open basis. The closed classes will have the form $\text{Mod}(T)$, where $T \subseteq I(\tau)$. If $=_L$ denotes $L$-elementary equivalence, let $S_\tau(L)$ be the quotient space $E_\tau(L)/=_L$, then the natural projection $\eta : E_\tau(L) \rightarrow S_\tau(L)$ establishes a lattice isomorphism between closed classes in $E_\tau(L)$ and closed classes in $S_\tau(L)$, and similarly for open classes. Therefore, properties such as compactness, normality, condition (ii) of Mundici’s theorem, etc., hold in $E_\tau(L)$ if and only if they hold in $S_\tau(L)$. For our purpose it is equivalent to work in one space or the other; however, it will be simpler and more natural to work in the first spaces. Since the logic is small, the topologies of these spaces are also small (they are indexed by sets) and there is no danger in working with them as with ordinary topological spaces. Of course, one could also make the spaces $E_\tau(L)$ small, by working with isomorphism classes of structures of cardinality no greater than the Löwenheim number of theories in $L(\tau)$, $\tau$ being the largest vocabulary involved in the arguments.

$E_\tau(L)$ is a completely regular space due to closure under negations, and so it is uniformizable by a standard result [10, Theorem 38.2]. In fact it has a canonical uniformity given by the uniformity basis:

$$U_\varphi = \{(A, B) \mid A \equiv_\varphi B\},$$

where $\varphi$ runs through the finite subsets of $L(\tau)$ and $\equiv_\varphi$ denotes equivalence of structures with respect to the sentences in $\varphi$. It is easy to see that this uniformity is totally bounded [10, Definition 39.7]. Analogous remarks hold for $S_\tau(L)$, which is moreover a Hausdorff space.

Given vocabularies $\tau^i$, $i \in I$, let $\tau^* = \bigcup_{i \in I} \{P_i\} \cup \tau'$ where the $P_i$ are new monadic predicates, and the $\tau'$ are mutually disjoint copies of the $\tau^i$. Then we may identify the Cartesian product $\Pi_{i \in I} E_{\tau^i}$ (which we will denote by $\Pi I E_{\tau^i}$ if the index set is understood) with the class of structures of type $\tau^*$ having the form

$$[A_i]_{i \in I} = \left(\bigcup_{i \in I} A_i, A_i\right), \quad A_i \in E_{\tau^i},$$

where the universe is the disjoint union of the universes of the $A_i$, each $P_i$ is interpreted by the disjoint copy $A_i$ of the universe of $A_i$, and $\tau'$ is interpreted in this universe by the corresponding copies of the relations of $A_i$. This is usually called the disjoint sum [5], or the full cardinal sum [1] of the structures $A_i$.

The product $\Pi I E_{\tau^i}$ inherits a topology as subspace of $E_{\tau^*}(L)$; it may be topologized also with the product topology of the $E_{\tau^*}(L)$. In general, both topologies differ. If the logic has relativizations, then the second topology is weaker than the first, because in such a case the relativized projections $\pi_i : E_{\tau^*}(L) \rightarrow E_{\tau^i}(L)$, $\pi_i(A) = A \upharpoonright \tau' \upharpoonright P^A$, are continuous, and when restricted to $\Pi I E_{\tau^i}$ become the ordinary projections of the product. It is not difficult to see that the product topology of $\Pi I E_{\tau^i}$ is generated by the subbasis of classes: $\text{Mod}(\phi^P) \cap \Pi I E_{\tau^i}$, $\phi_i \in L(\tau')$, where $\phi^P$ denotes the relativization of the sentence $\phi$ to the predicate $P$ (here we need the renaming property).
Although \( \prod_i E_{\tau^i} \) is not necessarily a closed subspace of \( E_{\tau^i}(L) \), we have the following property which will be useful.

**Lemma 1.** If \( M_1 \) and \( M_2 \) are disjoint closed subclasses in the product topology of \( \prod_i E_{\tau^i} \), then there are disjoint closed subclasses \( M_1, M_2 \) of \( E_{\tau^i}(L) \), defined by theories of power \( \sum_i |L(\tau_i)| \), such that \( M_j = M_j \cap \prod_i E_{\tau^i}, \ j = 1, 2 \).

**Proof.** Consider two disjoint closed classes \( M_1, M_2 \) in the product topology of \( \prod_i E_{\tau^i} \); then \( M_j = \text{Mod}(T_j) \cap \prod_i E_{\tau^i}, \ j = 1, 2 \), where \( T_j \) consists of finite disjunctions of relativized sentences \( \phi_i^{\tau^j}, \phi_i \in L(\tau^j) \). Let \( T^* \subseteq L(\tau^*) \) be the set of sentences stating that the interpretations of the \( P_i \) are disjoint, and each \( \tau^j \) is interpreted inside the interpretation of \( P_i \) (here we need the logic to contain some first-order formulae), and let \( M_j = \text{Mod}(T_j \cup T^*) \). Evidently \( \prod_i E_{\tau^i} \subseteq \text{Mod}(T^*) \) and so

\[
M_j \cap \prod_i E_{\tau^i} = \text{Mod}(T_j) \cap \text{Mod}(T^*) \cap \prod_i E_{\tau^i} = M_j.
\]

Moreover, \( M_1 \) and \( M_2 \) are disjoint, because if \( A \models T_j \cup T^* \), then \( A \models \bigcup_i P_i \approx [A \models P_i^A] \subseteq \prod_i, E_{\tau^i} \), and also \( A \models \bigcup_i P_i^A \subseteq T_j \), since each sentence of \( T_j \) depends only on some subuniverse \( P_i^A \subseteq \bigcup_i P_i \) of \( A \). Clearly, \( |T_j \cup T^*| \leq \sum_i |L(\tau^i)| + \omega \sum_i |\tau^i| = \sum_i |L(\tau^i)| \). □

**2. A property of uniform spaces**

Let \( \prod_i X_i = \prod_{i \in I} X_i \) be a Cartesian product of (perhaps proper) classes. We say that a subclass \( S \) of \( \prod_i X_i \) is of **finite index** if there are indices \( i_1, \ldots, i_n \in I \) such that whenever \( a \in S \) and \( \pi_{i_j}(a) = \pi_{i_j}(b) \) for \( j = 1, \ldots, n \), then \( b \in S \). This holds for example if \( S \) is a finite union of basics of a product topology in \( \prod_i X_i \). Recall that the **weight** of a topological space is the smallest power of its bases.

**Lemma 2.** Let \( X \) be a uniform noncomplete space. Then for any cardinal \( \kappa \geq \text{weight}(X) \) there are two disjoint closed subsets of the product space \( X^\kappa \) which cannot be separated by sets of finite index.

**Proof.** We assume the uniformity is given by a family \( G \) of pseudometrics. Let \( (a_\lambda)_{\lambda \in \Xi} \) be a Cauchy net in \( X \) having no limit, where \( \Sigma \) is a directed set which may be assumed to have power \( \text{weight}(X) \). For each \( (p, n) \in G \times \omega \) there is \( \mu(p, n) \) such that \( p(a_\lambda, a_\lambda') \leq 1/n \) for all \( \lambda, \lambda' \geq \mu(p, n) \). Let \( C_\lambda = \text{Cl}\{a_\eta; \beta \geq \lambda\} \), where \( \text{Cl} \) denotes the topological closure or adherence. Then \( \lambda \geq \lambda' \) implies \( C_{\lambda'} \subseteq C_{\lambda'} \), and \( C_\lambda \subseteq \text{Cl}_p\{a_\eta; \beta \geq \lambda\} \), the closure under the topology induced by the pseudometric \( p \), for any \( p \in G \). Therefore, if \( d_p \) denotes the diameter under \( p \), we have

\[
\lambda \geq \mu(p, n) \Rightarrow d_p(C_\lambda) \leq d_p(C_{\mu(p, n)}) \leq d_p(\text{Cl}_p\{a_\beta; \beta \geq \mu(p, n)\}) \leq \frac{1}{n}, \quad (1)
\]
the last inequality being true by the definition of \( \mu(p, n) \). Define now two subsets \( \mathcal{A} \) and \( \mathcal{B} \) of \( X \times X^2 \):

\[
\mathcal{A} = \left\{ (x, (x_\lambda)) \mid \forall (p, n) \in G \times \omega \forall \lambda \geq \mu(p, n) \colon p(x, x_\lambda) \leq \frac{1}{n} \right\},
\]

\[
\mathcal{B} = \left\{ (x, (x_\lambda)) \mid \forall \lambda \colon x_\lambda \in C_\lambda \right\} \quad \text{ (no condition on } x).\]

Both sets are closed by the continuity of the pseudometrics and the projections \( X^k \to X \). Moreover, \( \mathcal{A} \neq \emptyset \) (take \( x_\lambda = x \) constant), and \( \mathcal{B} \neq \emptyset \) since \( (x, (a_\lambda)) \in \mathcal{B} \) for any \( x \). To see that \( \mathcal{A} \cap \mathcal{B} = \emptyset \), assume \( (x, (x_\lambda)) \in \mathcal{A} \cap \mathcal{B} \), then \( p(x, x_\lambda) \leq 1/n \) and \( x_\lambda \in C_\lambda \) for any \( (p, n) \in G \times \omega, \lambda \geq \mu(p, n) \). By (1) this implies

\[
p(x, a_\lambda) \leq \frac{2}{n},
\]

for all \( (p, n) \in G \times \omega, \lambda \geq \mu(p, n) \); which in turn implies \( a_\lambda \to x \), a contradiction.

Finally, we show that \( \mathcal{A} \) and \( \mathcal{B} \) may not be separated by subsets of finite index of \( X^k \). Assume \( U \) is of finite index and \( \mathcal{B} \subseteq U \). Then, except for finitely many indexes \( \lambda_1, \ldots, \lambda_m \), the \( \lambda \)-components of any \( (x, (x_\lambda)) \in U \) are arbitrary. Choose \( \lambda_0 \geq \lambda_1, \ldots, \lambda_m \) and define

\[
b_\lambda = \begin{cases} a_{\lambda_0}, & \text{if } \lambda \neq \lambda_1, \ldots, \lambda_m, \\ a_\lambda, & \text{otherwise}; \end{cases}
\]

then \( (a_{\lambda_0}, (b_\lambda)) \in \mathcal{B} \subseteq U \) and so \( (a_{\lambda_0}, (b_\lambda)) \in U \). Moreover, for any \( (p, n) \in G \times \omega \) and \( \lambda \geq \mu(p, n) \) we have \( p(a_{\lambda_0}, b_\lambda) \leq 1/n \). If \( \lambda \neq \lambda_1, \ldots, \lambda_m \), because \( p(a_{\lambda_0}, b_\lambda) = 0 \), and if \( \lambda = \lambda_i \) for some \( i = 1, \ldots, m \), because then \( \lambda_0 \geq \lambda \geq \mu(p, n) \) and so \( p(a_{\lambda_0}, b_\lambda) = p(a_{\lambda_0}, a_\lambda) \leq 1/n \). Therefore, \( (a_{\lambda_0}, (b_\lambda)) \in \mathcal{A} \), which shows that \( U \) does not separate \( \mathcal{A} \) and \( \mathcal{B} \). Taking \( \mathcal{A} \times X^k \), \( \mathcal{B} \times X^k \) in \( X \times X^2 \times X^k = X^k \) for \( k \leq |\Sigma| = \text{weight}(X) \), we obtain two closed subsets of \( X^k \) inseparable by sets of finite index.

It is not difficult to see that the above proposition applies to large spaces (those where the domain is a proper class) with small topologies.

A subclass \( S \) of \( E_\tau \) will be said to be of finite dependence if there is finite \( \mu \subseteq \tau \) such that if \( A \in S \) and \( B \mid \mu = A \mid \mu \), then \( B \in S \).

**Theorem 3.** A small logic \( L \) is compact if and only if for any \( \tau \) in the domain of \( L \), any pair of disjoint closed classes in \( E_\tau(L) \) are separable by a class of finite dependence.

**Proof.** Follows from the next more general theorem. □

Recall that a logic \( L \) has occurrence number \( \alpha \) if \( \alpha \) is the smallest cardinal such that \( L(\tau) = \bigcup \{ L(\mu) \colon \mu \subseteq \tau, |\mu| < \alpha \} \) for any \( \tau \) in its domain (cf. [5, 2.1.4]).
Theorem 4. A logic $L$ with occurrence number at most $\kappa^+$ is $\kappa$-compact if and only if any pair of closed classes of $E_\tau(L)$ defined by jointly unsatisfiable theories $T_1, T_2$ of $L$ of power $\kappa$ may be separated by a class of finite dependence.

Proof. If the logic is $\kappa$-compact, it is obvious that the separation of closed classes given by theories of power $\kappa$ may be achieved by an elementary class; and using that each sentence has at most $\kappa$ symbols, plus $\kappa$-compactness, any elementary class has finite dependence [2, 5.1.2]. Now, let $T \subseteq L(\tau)$ be a counterexample to $\kappa$-compactness. We may assume $|L(\tau)| = \kappa$, by taking an appropriate sublogic, and $|\tau| = \kappa$, by the hypothesis on the occurrence number. As the space $E_\tau(L)$ is not compact, it is not complete as uniform space (being totally bounded [10, Theorem 39.9]). By Lemma 2, there are disjoint closed subclasses $M_1, M_2$ in the product topology of $E_\tau(L)^\kappa$ inseparably by classes of finite index. Then the disjoint closed subclasses $M_1, M_2$ of $E_\tau(L)$ provided by Lemma 1 must be inseparable by classes of finite dependence; otherwise the restrictions of the separating classes to $E_\tau(L)$ would be of finite index and would separate the $M_i$. Moreover, $M_1$ and $M_2$ are defined by theories of power $\Sigma_{i<\kappa} |L(\tau)| = \kappa$. □

Remark 5. It is easy to see that for the validity of Theorem 3 we do not need the logic to be defined in all vocabularies: we only need closure of the vocabularies under the operation $\tau \mapsto \bigcup_{\alpha<|L(\tau)|} \{P_\alpha\} \cup \exists^\alpha$, where the $\exists^\alpha$ are mutually disjoint copies of $\tau$, because the $\kappa$ of Lemma 2 may be taken to be $|L(\tau)|$. Similarly, in Theorem 4 we only need closure under $\tau \mapsto \bigcup_{\alpha<\kappa} \{P_\alpha\} \cup \exists^\alpha$. For example, these theorems hold for monadic logics. The same is true for the other results in this paper.

Notice that to achieve compactness in Theorems 3 and 4 we do not need the separating class to be clopen as in [6], but just to have finite dependence. Separation of closed classes by clopen or elementary classes without the finite-dependence hypothesis is not enough to yield compactness, at least in the case when there are large cardinals, as the following example illustrates.

Example 6. For a compact cardinal $\kappa$, the logic $L_{\kappa\kappa}$ is not compact but it is $(\infty, \kappa)$-compact; hence, any two mutually unsatisfiable theories $T_1, T_2$ of $L_{\kappa\kappa}$ are separable by a sentence: find unsatisfiable $S \subseteq T_1 \cup T_2$ of power less than $\kappa$, then $\bigwedge (S \cap T_i)$ provides the separating sentence. Similarly, if $\kappa$ is measurable, then $L_{\kappa\kappa}$ has occurrence number $\kappa$, and by $(\kappa, \kappa)$-compactness any pair of mutually unsatisfiable theories of power $\kappa$ are separable by a sentence, but the logic is not $\kappa$-compact.

Corollary 7. $L = L(Q^i : i \in \omega)$ is $\omega$-compact if and only if for countable $\tau$ any clopen of $E_\tau(L)$ is $L$-elementary.
Proof. One direction is trivial. For the other notice that for countable τ the space $E_τ(L)$ has a countable basis; hence, it is a Lindelöff regular space and so it is normal by [10, Theorem 16.8]. As it has a basis of clopen classes, the separation of closed classes may be achieved by a clopen class by [4, Theorem 16.16]. By hypothesis, this class is elementary and of finite dependence. Then apply Theorem 4 with $κ = ω$. □

The following simple but nice consequence of Theorem 3 was proved by Väänänen for countably generated nonmonadic logics [9, Remark 4]. Our proof does not need those assumptions. Let Bethₙ mean Beth's definability theorem with the implicit definition of the predicate $R$ given by a theory $T(R) ⊆ L(τ)$ instead of a single sentence, and the explicit definition of $R$ given by a sentence of $L(τ)$. Beth will mean the ordinary Beth's theorem. A logic $L$ will have finite dependence number if $Mod(φ)$ is of finite dependence for any sentence $φ$ of $L$.

Corollary 8. For any small logic with finite dependence number and closed under $∀$, Bethₙ = Beth + compactness.

Proof. That Beth + compactness implies Bethₙ is straightforward. To show that Bethₙ implies compactness, assume $T_1, T_2 ⊆ L(τ)$ are jointly unsatisfiable theories and consider the following implicit definition for the new monadic symbol $R$:

$$\left( \bigwedge T_1 \land ∀x \, R(x) \right) \lor \left( \bigwedge T_2 \land ∀x \, ¬R(x) \right).$$

The above disjunction between two arbitrary conjunctions is equivalent to a single theory $T(R)$ since the union of closed sets is closed; moreover, it defines $R$ implicitly as being the universe or the empty set. Find an explicit definition $θ(x) ∈ L(τ)$ of $R$; then

$$A ⊨ T_1 \implies (A, A) ⊨ T(R) \implies (A, A) ⊨ ∀x \, (R(x) ↔ θ(x)) \implies A ⊨ ∀x \, θ(x).$$

Similarly,

$$A ⊨ T_2 \implies (A, ∅) ⊨ ∀x \, (R(x) ↔ θ(x)) \implies A ⊨ ∀x \, ¬θ(x) \implies A ⊭ ∀x \, θ(x);$$

hence, $∀x \, θ(x)$ separates $T_1$ and $T_2$. By finite dependence of $∀x \, θ(x)$ and Theorem 3, the logic is compact. □

3. Uniform reduction and the product topology

As we have noticed in Section 1, the topology inherited by $Π_t E_τ$ as subspace of $E_τ(L)$ does not necessarily coincide with the product topology of the $E_τ(L)$. However, both topologies are identical under very natural conditions.
**Definition 9.** A logic $L$ has the weak uniform reduction property for $\kappa$-sums, if given $\{\tau^i\}_{i \in I}$ with $|I| = \kappa$, then any $\theta \in L(\tau^*)$ is equivalent in all sums $[A_i]_{i \in I} \in \prod_{i \in I} E_{\tau^i}$ to the disjunction of a (possibly infinite) family of formulae $\{\phi_i\}_{i \in I}$, where each $\phi_i$ is a finite Boolean combination of relativized formulae $\psi^0, \psi \in L(\tau^i)$ for some $i \in I$. If the disjunction may be taken always finite, we drop the adjective weak.

The uniform reduction property for 2-sums is equivalent to the familiar uniform reduction property for pairs, URP in short, cf. [5, Definition 4.2.9], which we may express as follows: given $\theta \in L(\{P_1\} \cup \tau^1 \cup \{P_2\} \cup \tau^2)$, there are sentences $\phi_1, \ldots, \phi_n \in L(\tau^1)$, and $\psi_1, \ldots, \psi_n \in L(\tau^2)$ such that for any $[A_1, A_2] \in E_{\tau^1} \times E_{\tau^2}$:

$$[A_1, A_2] \vDash \theta \text{ if and only if } \bigvee (A_1 \vDash \phi_i \land A_2 \vDash \psi_i).$$

The weak uniform reduction property for $\kappa$-sums implies the Feferman–Vaught property for $\kappa$-sums: if $A_n =_L B_n$ for all $\alpha < \kappa$, then $[A_n]_{\alpha < \kappa} =_L [B_n]_{\alpha < \kappa}$. For compact logics, uniform reduction, weak uniform reduction and the Feferman–Vaught property for $\kappa$-sums become equivalent. The following obvious fact justifies our interest in these properties.

**Fact 10.** $L$ has the weak uniform reduction property for $\kappa$-sums if and only if, whenever $|I| = \kappa$, the subspace topology induced by $E_{\tau^i}(L)$ in $\prod_{i \in I} E_{\tau^i}$ is the product topology of the spaces $E_{\tau^i}(L)$.

By induction, (weak) URP implies (weak) uniform reduction for all finite sums. In some case, URP implies uniform reduction for large enough infinite sums.

**Proposition 11.** Let $L$ have a finite dependence number and assume that $L$ does not distinguish cardinals above $\beta$; this is $(\kappa) =_L (\kappa')$ for $\kappa, \kappa' \geq \beta$. If $L$ has (weak) URP, then it has (weak) uniform reduction for $\kappa$-sums, for any $\kappa \geq \beta$.

**Proof.** Fix $\{\tau^i: i \in I\}$. Given $\theta \in L(\tau^*)$, the finite occurrence property implies that there are indices $i_1, \ldots, i_n \in I$ such that for any disjoint sum $[A_i]_{i \in I}$ then

$$[A_i]_{i \in I} = \left( \bigcup A_i, \bigcup A_i \right)_{i \in I} \vDash \theta \text{ iff } [A_{i_1}, \ldots, A_{i_n}, (X)] \vDash \theta,$$

where $X = \bigcup \{A_j: j \neq i_1, \ldots, i_n\}$, and $(X)$ is a structure of empty type. By the URP, this is in turn equivalent to

$$[A_{i_1}, \ldots, A_{i_n}, (X)] \vDash \bigvee \theta_r,$$

with $\theta_r = \phi^0_{r, i_1} \land \cdots \land \phi^0_{r, i_n} \land \psi^Q_r$ for certain sentences $\phi^0_{r, i} \in L(\tau^i), \psi_r \in L(\theta)$, where $Q$ is interpreted by $X$. Now, by hypothesis, $\psi_r$ has the same truth value in
all sets of cardinality at least \( \beta \), and certainly \(|X| \geq \beta \) if \(|I| \geq \beta \). We have then for \(|I| = \kappa \geq \beta \) summands:

\[
[A_i]_{i \in I} \models \theta \iff \bigvee_{s} \theta^*_s \iff \bigwedge_{s} \theta^*_s,
\]

where \( s \) runs over the indices \( r \) for which \( \psi_r \) is true in all sets of power at least \( \beta \) (the empty disjunction is taken as falsity), and \( \theta^*_s \) results of changing \( \psi^Q_2 \) to \( \forall x \ (x = x) \).

**Example 12.** (a) First-order logic has uniform reduction for all sums. This follows from [3], but may be proved also using partial isomorphisms.

(d) If \( L(Q^i : i \in I) \) satisfies URP and the downward L"owenheim–Skolem theorem for sentences, then it has uniform reduction for all sums, because then \( L \) satisfies the hypothesis of Proposition 11 for \( \beta = \omega \). This is the case of \( L_{\omega_1\omega} \) and \( L(Q_0) \).

(c) \( L_{\omega_1\omega}(Q_\alpha : \alpha \in S) \) has uniform reduction for \( \kappa \)-sums if \( \kappa \geq \beta = \sup\{\omega_\alpha : \alpha \in S\} \). Utilizing back-and-forth methods one may show that this logic has URP, and by the downward L"owenheim–Skolem theorem for theories, it is easy to see that \( \beta \) satisfies the hypothesis of Proposition 11.

### 4. Compactness and normality

Call a small logic \( L \) normal if each \( E_\tau(L) \) is a normal topological space. More generally, we say that \( L \) is \( \kappa \)-normal if disjoint closed classes defined by theories of \( L \) of power at most \( \kappa \) may be separated by open classes. Evidently, normality (respectively, \( \kappa \)-normality) may be expressed by the following semantical property: given theories (respectively, theories of power \( \kappa \)) \( T_1, T_2 \) in \( L(\tau) \) such that \( \bigwedge T_1 \models \bigvee T_2 \) there are theories \( T_3, T_4 \) in \( L(\tau) \) such that

\[
\bigwedge T_1 \models \bigvee T_3 \models \bigwedge T_4 \models \bigvee T_2.
\]

**Example 13.** (a) Any logic \( L \) is countably normal. Given countable mutually inconsistent theories \( T_1, T_2 \) of \( L(\tau) \) find a countable subset \( M \) of \( L(\tau) \) containing \( T_1, T_2 \) and closed under finite Boolean operations; then \( E_\tau(M) \) is a regular Lindelöf space (has a countable basis). Hence, Mod(\( T_1 \)) and Mod(\( T_2 \)) may be separated by a clopen class by [4, Theorem 16.16].

(b) Any compact logic is normal; similarly, \( \kappa \)-compactness implies \( \kappa \)-normality. Trivial.

(c) If \( \kappa \) is a compact cardinal, then \( L_{\kappa\kappa} \) is normal. If \( \kappa \) is weakly compact, \( L_{\kappa\kappa} \) is \( \kappa \)-normal. See Example 6.

Since Mundici’s theorem says that a small logic is compact if it is normal in a strong sense, it is natural to ask whether plain normality is enough to yield
compactness of a logic. If there are large cardinals, the answer is negative, by Example 13(c). However, utilizing the following topological result due to Noble, we show that it is positive whenever the logic has weak uniform reduction for all disjoint sums.

**Lemma 14** (Noble [7, Corollary 2.2], Przymusiński [8, Corollary 6.6]). A Hausdorff space $X$ is compact if and only if for all $\kappa$ (or just for some uncountable $\kappa \geq \text{weight}(X)$) the product space $X^\kappa$ is normal.

**Theorem 15.** Let $L$ be a small logic having the weak uniform reduction property for $\kappa$-sums, for arbitrarily large $\kappa$; then $L$ is compact if and only if $L$ is normal.

**Proof.** One direction is trivial. For the other, assume $L$ is not compact; then $S_\tau(L)$ is a noncompact Hausdorff space for some $\tau$, and by Lemma 14, $S_\tau(L)^\kappa$ is a nonnormal product space for any uncountable $\kappa \geq |L(\tau)|$. It must have then two closed subsets inseparable by open sets. Pulling back the counterexample through the natural projection $\eta^*: E_\tau(L)^\kappa \to S_\tau(L)^\kappa$, we obtain two closed subclasses $M_1$, $M_2$ of $E_\tau(L)^\kappa$ inseparable by open classes (all in the product topology). Choose $\kappa$ so that $L$ has the weak uniform reduction property for $\kappa$-sums; then $M_1$ and $M_2$ are inseparable in the subspace topology, and by Lemma 1 we conclude that $E_\tau(L)$ is not normal. \[\square\]

As any logic is $\omega$-normal (Example 13(a)), the previous theorem cannot be generalized to say that, under the hypothesis of uniform reduction for disjoint sums, $\kappa$-normality yields $\kappa$-compactness. However, $\kappa = \omega$ is the only exception to this statement, at least for logics of the form $L(Q^i: i \in I)$, $|I| \leq \kappa$, because of the next theorem.

**Theorem 16.** Let $L$ be a logic with occurrence number at most $\kappa^+$ for some uncountable $\kappa$, and such that $|L(\tau)| = \kappa$ whenever $|\tau| = \kappa$. If $L$ has weak uniform reduction for $\kappa$-sums, then $L$ is $\kappa$-compact if and only if it is $\kappa$-normal.

**Proof.** If $L$ is not $\kappa$-compact, then for some $\tau$, $S_\tau(L)$ is not compact, where by the hypothesis about the occurrence number, $\tau$ may be assumed to have power $\kappa$. Proceedings as in the proof of Theorem 15 we get that $E_\tau(L)$ is not normal, and because $|\tau^*| = |L(\tau)| |\tau| = \kappa = |L(\tau^*)|$, $L$ is not $\kappa$-normal. \[\square\]

**Example 17.** From Examples 12(b), (c) and Theorem 16 we have:

(a) $L(Q_\emptyset)$ is not $\omega_1$-normal even in monadic vocabularies, because it has uniform reduction for all sums, but it is not $\omega_1$-compact.

(b) If $S \neq \{0\}$, then $L(Q_\alpha: \alpha \in S)$ is not $\kappa$-normal even in monadic vocabularies for $\kappa = \sup\{\omega_\alpha: \alpha \in S\}$, because it has uniform reduction for $\kappa$-sums, but it is not $\kappa$-compact.
Corollary 18. Let $\kappa$ be an uncountable cardinal. If $L = L(Q^i : i \in I)$, $|I| \leq \kappa$, has uniform reduction for $\kappa$-sums, and is $(\kappa, \omega_1)$-compact, then it is $\kappa$-compact.

Proof. Let $T_1, T_2 \subseteq L(\tau)$ be mutually unsatisfiable theories of power $\kappa$. We may assume $|\tau| = \kappa$. Then $E_{\kappa}(L)$ is a Lindelöf space by $(\kappa, \omega_1)$-compactness. As a Lindelöf regular space is normal [10, theorem 16.8], the result follows from Theorem 16. □

Corollary 19 (à la Lindström). Let $L$ be a small logic with finite-dependence number satisfying the downward Löwenheim–Skolem theorem for sentences, uniform reduction for pairs, and normality; then $L = L_{\omega_1}$.

Proof. Under the hypothesis, $L$ has uniform reduction for all sums by Proposition 11, and it must be compact by Theorem 15; then apply the first Lindström theorem. □

If $L$ has the form $L(Q^i : i \in I)$, $|I| \leq \omega_1$, then the normality hypothesis in Corollary 19 may be weakened to $\omega_1$-normality, which yields $\omega_1$-compactness by Theorem 16, enough to apply Lindström’s theorem.

References