Area exam: geometry and topology.

The program consists of basic topics indicated by (B) and advanced topics (A). When preparing the exam, we recommend to study first the basic topics, and then pass to the advanced topics. The exam includes seven exercises, four taken from the basic topics, and three from advanced topics. In order to get maximal grade it is sufficient to give a complete solution to five exercises out of the seven ones.

Exam topics:

1. Topology (B)

(1) Definition of topological space. Basis and subbasis for topology. ([JRM], Ch. 2, 12–13.) Examples: order topology, product topology, subspace topology. ([JRM], Ch. 2, 14–16.)
(2) Closed sets and limit points. Hausdorff spaces. ([JRM], Ch. 2, 17.)
(3) Continuous functions and their properties. Homeomorphism. ([JRM], Ch. 2, 18.)
(4) The product topology. ([JRM], Ch. 2, 19.)
(5) The metric topology. ([JRM], Ch. 2, 20–21.)
(6) The quotient topology. ([JRM], Ch. 2, 22.)
(7) Connected topological spaces. ([JRM], Ch. 3, 23–25.)
(8) Compact and topological spaces. Tikhonov theorem. ([JRM], Ch. 3, 26–28; Ch. 5, 37.)
(9) Locally compact topological spaces. One point compactification. Čhech-Stone compactification. ([JRM], Ch. 3, 29; Ch. 5, 38.)
(10) Countability axioms. ([JRM], Ch. 4, 30).
(11) Separation axioms. ([JRM], Ch. 4, 31).
(12) Topological manifolds, classification of one-dimensional topological manifolds and two-dimensional compact manifolds. ([WSM], Chapter 1).

2. Algebraic topology (B)

(1) Homotopy of continuous maps. Homotopy equivalence. ([AT], Ch.0).
(2) CW-complexes. ([AT], Ch.0).
(3) Euler Characteristic ([AT], Ch.2).,
(4) Fundamental group: the functor π_1 from the category of topological spaces to the category of groups; fundamental group of a circle; van-Kampen's theorem; fundamental group of two-dimensional compact manifolds; retracts and deformational retracts; applications: Brouwer fixed point theorem; Borsuk-Ulam theorem; fundamental theorem of algebra. ([AT], Ch.1, 1.1–1.2).
(5) Covering spaces. Lifting properties. Deck transformations. The classification of covering spaces. ([AT], Ch.1, 1.1–1.2).
(6) Simplicial and singular Homology. Simplicial complex, singular complex. Homotopy Invariance of singular homology. ([AT], Ch.2, 2.1).
(7) Relative homology. Long exact homology sequence of a good pair. Excision theorem. The equivalence of simplicial and singular homology. ([AT], Ch.2, 2.1).
(8) Mayer-Vietoris sequences. ([AT], Ch.2, 2.2).
(9) Degree of a continuous map. Calculation homology of CW-complexes. ([AT], Ch.2, 2.2).
(10) Homology with Coefficients. Axioms of homology functor. ([AT], Ch.2, 2.2–2.3).
(11) Relation between $\pi_1(X)$ and $H_1(X;\mathbb{Z})$. ([AT], Ch.2, 2A).
(12) Applications of homology: Generalizations of Jordan curve theorem, invariance domain theorem, ([AT], Ch.2, 2B).
(13) Cohomology of algebraic complex. Universal coefficient theorem. Axioms of cohomology. ([AT], Ch.3, 3.1).
(14) Simplicial, singular, CW (relative) cohomology. Homotopy invariance. Mayer-Vietoris sequences. ([AT], Ch.3, 3.1).

(15) Cup product. Cohomology ring. ([AT], Ch.3, 3.2).
(16) Orientation of manifolds ([AT], Ch.3, 3.2).
(17) The Künneth formula. ([AT], Ch.3, 3.2).
(18) Poincaré duality. ([AT], Ch.3, 3.3).
(19) Topology of Riemann surfaces: genus, Euler characteristic, singular cohomology groups, fundamental group.

3. Geometry of manifolds and Lie groups **(B)**

(1) Differentiable manifold structure. Differentiable maps of differentiable manifolds. Diffeomorphism. Example of differentiable manifolds: $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, compact surfaces, Grassmann manifolds, matrix groups. ([FW], Ch. 1, 1.1–1.6).
(2) Partition of unity. ([FW], Ch. 1, 1.7–1.11).
(3) Tangent space of a manifold. Definitions of tangent vectors: classes of equivalent curves, differential operators. Differential of a smooth map. Equivalence of these definitions in the smooth case. ([FW], Ch. 1, 1.12–1.26).
(4) Immersions, embeddings. Submanifolds. Implicit function theorems. ([FW], Ch. 1, 1.27–1.40).
(5) Vector fields. The flow of a vector field. The Lie bracket of vector fields. ([FW], Ch. 1, 1.41–1.55).
(6) Distributions and Frobenius theorem. ([FW], Ch. 1, 1.56–1.64).
(7) Tensor and exterior algebras. ([FW], Ch. 2, 2.1–2.13).
(8) Tangent bundle. Cotangent bundle. Tensor bundle. Tensor fields and differential forms. ([FW], Ch. 2, 2.14–2.23).
(9) Lie derivative. ([FW], Ch. 2, 2.24–2.25).
(10) Differential ideals. ([FW], Ch. 2, 2.26–2.34).
(11) De Rham cohomology. De Rham cohomology with compact support. ([BT], Ch. 1, 1).
(12) Mayer-Vietoris sequence for de Rham cohomology (also with compact support). ([BT], Ch. 1, 2).
(13) Integration of differential forms on manifolds. The Stokes theorem. ([BT], Ch. 1, 3).
(14) Poincaré lemmas. Homotopy invariance for de Rham cohomology. ([BT], Ch. 1, 4).
(15) Generalized Mayer-Vietoris principle for de Rham cohomology. Čech cohomology. ([BT], Ch. 2, 8–10).
(16) Lie group. Lie algebra of a Lie group. ([FW], Ch. 3, 3.1–3.16).
(17) Action of a Lie group on a manifold. Homogeneous manifolds. ([FW], Ch. 3, 3.58–3.68).

4. Fiber Bundles **(B, A)**

(1) **(B)** Principal bundles. Associated bundles. ([KN], Ch. 1, 5).
(2) **(B)** Vector bundles. Operations on vector bundles. ([KN], Ch. 1, 5; [BT], Ch. 1, 6).
(3) **(B)** Connection in a principle bundle. Holonomy group. ([KN], Ch. 2, 1–3).
(4) **(B)** Curvature form. The structure equations. ([KN], Ch. 2, 5).
(5) **(A)** Reduction theorem. ([KN], Ch. 2, 6–8).
(6) **(B)** Linear connections in vector bundles. ([KN], Ch. 3, 1).
(7) **(B)** Linear connections in tangent bundles. ([KN], Ch. 3, 2).
(8) **(B)** Curvature and torsion tensors. ([KN], Ch. 3, 5, 7).
(9) **(B)** Geodesics. ([KN], Ch. 6, 7).
(10) **(A)** Normal coordinates. Linear infinitesimal holonomy group.

5. Riemannian Geometry **(B, A)**

(1) **(B)** Riemannian connections. Riemannian metrics and Riemannian connections. Completeness. ([KN], Ch. 4, 1–4).

(2) **(A)** Holonomy groups. The decomposition theorem of de Rham. ([KN], Ch. 4, 5–6).
(3) **(B)** Curvature and space forms. Sectional curvature. Spaces of constant curvature. ([KN], Ch. 5, 1–3).
(4) **(A)** Flat linear and Riemannian connections. ([KN], Ch. 5, 4).
(5) **(B)** Submanifolds. The Gauss map. Covariant differentiation and second fundamental form. Equations of Gauss and Codazzi. ([KN2], Ch. 7, 1–4).
(6) **(B)** Variations of length integral. Jacobi fields. Conjugate points. Comparison theorems. ([KN2], Ch. 8, 1–4).
(7) **(A)** The first and second variations of length integral. Index theorem of Morse. Cut loci. Spaces of non-positive curvature. ([KN2], Ch. 8, 5–8).

6. Differential Topology **(A)**

(1) Sard's theorem ([GP], Ch. 1,7).
(2) Morse functions ([GP], Ch. 1,7;[M], Ch. 1,1-4).
(3) Transversality ([GP], Ch. 1,5-6).
(4) Lefschetz fixed point theory. ([GP], Ch. 3,4).
(5) Vector Fields and the Poincare-Hopf theorem. ([GP], Ch. 3,5) ([MT], Ch. 11-12).
(6) Gauss-Bonnet ([GP], Ch. 4,9).

7. Characteristic Classes and geometry of differential forms **(A)**

(1) Cohomology of $BO(n)$ and $BU(n)$([M2], Ch. 7,14) ([MT], Ch. 20).
(2) Thom isomorphism ([M2], Ch. 10) ([BT], Ch. 1, 6) ([MT], Ch. 21).
(3) Stiefel-Whitney Classes ([M2], Ch. 4,7-8)
(4) Euler Class ([M2], Ch. 9) ([BT], Ch. 6, 11) ([KN2], Ch 12).
(5) Chern Classes ([M2], Ch. 14) ([BT], Ch. 20) ([MT], Ch. 19) ([KN2], Ch 12).
(6) Pontryagin Classes ([M2], Ch. 15) ([BT], Ch. 22) ([MT], Ch. 18) ([KN2], Ch. 12). ([BT], Ch. 1, 6; [KN2], Ch. 12).
(7) Chern-Weyl theory ([KN2], Ch 12), Appendix C) ([MT], Ch. 17-21).

8. Analysis on manifolds. Hodge theorem. **A**

(1) The Laplace-Beltrami operator. ([FW], Ch. 6, 6.1–6.3).
(2) Hodge decomposition theorem. ([FW], Ch. 6, 6.4–6.14).
(3) Sobolev spaces. ([FW], Ch. 6, 6.15–6.23).
(4) Linear differential operators. ([FW], Ch. 6, 6.24–6.27).
(5) Elliptic operators. ([FW], Ch. 6, 6.28–6.30).
(6) Regularity theorem. ([FW], Ch. 6, 6.4–6.6, 6.31–6.32).
(7) Ellipticity of Laplace-Beltrami operator. ([FW], Ch. 6, 6.34–6.36).

9. Riemann surfaces ([FO],[FA]) **A**

(1) Holomorphic functions and analytic functions; Cauchy integral formula.
(2) Cauchy-Riemann equations.
(3) Maximum principle, open mapping theorem.
(4) Meromorphic functions.
(5) The simply connected Riemann surfaces (the Riemann sphere, the complex plane, and the upper-half plane).
(6) Automorphism groups of simply connected Riemann surfaces.
(7) The hyperbolic structure of the upper-half plane.
(8) Degree of a holomorphic mapping. Relation with covering spaces.
(9) Riemann-Hurwitz formula

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[FO] Forster, Lectures on Riemann surfaces. Chapter 1, Sections 1 to 10.
[FA] Farkas-Kra, Riemann surfaces. Chapters 0 and 1.