A FORMAL SYSTEM FOR THE NON-THEOREMS OF THE PROPOSITIONAL CALCULUS

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Introduction The completeness of the classical propositional calculus allows us to give a deductive system consisting of finitely many axiom schemas and finitely many rules of inference, that permit us to pass from a formula or a pair of formulae to a syntactically related formula, in such a manner that the formulae obtained inductively from the axioms by repeated application of the rules are exactly the tautologies. In this paper we give an analogous deductive system (more concretely, a Hilbert type system) such that the formulae deduced are exactly those that are not tautologies, the non-theorems of the propositional calculus. Obviously, this has to be the most non-standard of the non-classical logics. It is important to note that there are many other algorithms to generate recursively the non-theorems, since the propositional calculus is decidable. Usually they are based in the methodical search for a counterexample, but they lack the inductive character of a Hilbert type system, where every formula involved in a deduction is itself deductible. In our system, unlike semantic tableaux or refutation trees, every formula introduced in a deduction is a non-tautology, and it is introduced only if it is a non-tautological axiom, or it follows by one of the non-tautological rules of inference from non-tautologies introduced earlier in the deduction.

1 Axioms and rules We assume that the only connectives are ~ and ⊃. p, q, p1, p2, . . . denote atomic formulae. α, β, γ, . . . denote arbitrary formulae. We define ρ(α) = {p | p occurs in α}.

Axioms

A1 p ⊃ ~p (p atomic)
A2 ~p ⊃ p (p atomic)

Rules

R1 (a) \[
\frac{\alpha}{p \supset \alpha} \quad (p \text{ atomic, } p \text{ does not occur in } \alpha)
\]
R1 (b) \[ \frac{\alpha}{\sim \beta \supset \alpha} \] (\( \beta \) atomic, \( \beta \) does not occur in \( \alpha \))

R2 \[ \frac{\alpha \supset \beta}{\alpha \supset (\beta \supset \alpha)} \]

R3 \[ \frac{\alpha \supset \beta}{(\gamma \supset \alpha) \supset \beta} \]

R4 \[ \frac{\sim \alpha \supset \beta}{(\alpha \supset \gamma) \supset \beta} \]

R5 \[ \frac{\sim \alpha \supset \beta}{\alpha} \]

R6 \[ \frac{\alpha \supset \beta}{\sim \sim \alpha \supset \beta} \]

R7 \[ \frac{\alpha \supset (\beta \supset \gamma)}{\beta \supset (\alpha \supset \gamma)} \]

R8 \[ \frac{\alpha \supset S, \sim \beta \supset S}{\sim (\alpha \supset \beta) \supset S} \] (where \( S \) has the form indicated below)

The formula \( S \) in R8 must have the form \( S = S_i \) or \( S = S_i \supset (S_{i+1} \supset \ldots (S_{n-1} \supset S_n) \ldots) \), with \( S_i = p_i \) or \( S_i = \sim p_i \), \( p_i \neq p_j \) for \( i \neq j \), and \( \mathcal{P}(\alpha \supset \beta) \subseteq \{p_1, p_2, \ldots, p_n\} \).

Note that the axioms cannot be replaced with schemata, and a substitution rule cannot be allowed, since many non-tautologies become tautologies through substitution. We use the notation \( \vdash \alpha \) to indicate that the formula \( \alpha \) is deducible in the above system.

Examples

1. \( \vdash (p \supset q) \supset (q \supset p) \)
   1. \( \sim p \supset p \) A2
   2. \( p \) R5
   3. \( q \supset p \) R1
   4. \( q \supset (q \supset p) \) R2
   5. \( (p \supset q) \supset (q \supset p) \) R3

2. \( \vdash \sim p \)
   1. \( p \supset \sim p \) A1
   2. \( \sim \sim p \supset \sim p \) R6
   3. \( \sim p \) R5

3. \( \vdash \sim (p \supset p) \)
   1. \( p \supset \sim p \) A1
   2. \( (p \supset p) \supset \sim p \) R3
   3. \( \sim (p \supset p) \supset \sim p \) R6
   4. \( \sim (p \supset p) \) R5

4. \( \vdash ((p \supset \sim p) \supset (\sim q \supset q)) \supset q \)

We give the "proof" in tree form, since in this example the use of R8 seems essential:
### Completeness

As usual, $\vdash \alpha$ means that $\alpha$ is a tautology. We show that our system is perfectly unsound and completely antitautological. In other words, we prove the following:

**Theorem A.** If $\vdash \alpha$ then not $\vdash \alpha$.

B. If not $\vdash \alpha$ then $\vdash \neg \alpha$.

**Proof:**

A. We use the symbol $\# \alpha$ to indicate that there is a valuation $\nu$ such that $\nu(\alpha) = F$. It is clear that $\# p \supset \neg p$ and $\# \neg p \supset p$, for $p$ atomic, and rules R1 to R7 preserve this property; in fact, R2, R6, and R7 are logical equivalences and preserve "everything". The only non-trivial case is that of rule R8. Let $S$ be as explained in the rule, and let $\nu$ and $\omega$ be valuations such that $\nu(\alpha \supset S) = F$ and $\omega(\neg \beta \supset S) = F$. Then $\nu(S) = \omega(S) = F$ and so: $\nu(S_i) = \omega(S_i) = T$ for $i < n$, $\nu(S_n) = \omega(S_n) = F$. But these conditions determine completely the valuations in $p_1, p_2, \ldots, p_n$, thus $\nu \{p_1, p_2, \ldots, p_n\} = \nu(S) = \omega(S) = F$. Since $P(\alpha) \cup P(\beta) \subseteq \{p_1, p_2, \ldots, p_n\}$, we have $\nu(\alpha) = \nu(\alpha) = T, \nu(\beta) = \omega(\beta) = F, \nu(S) = \omega(S) = F$, and so $\nu(\neg (\alpha \supset \beta) \supset S) = F$. This finish the proof.

B. We prove first, by induction in the complexity of the formula $\alpha$, the following property:

\[ (*) \]

\[
\begin{align*}
\{& \text{If } P(\alpha) \subseteq \{p_1, p_2, \ldots, p_n\}, S = S_1 \supset (S_2 \supset \ldots (S_{n-1} \supset S_n) \ldots) \text{ with } p_i \neq p_j \text{ for } i \neq j, \text{ and } S_i = p_i \text{ or } S_i = \neg p_i, \text{ then: } \# \alpha \supset S \text{ implies } \vdash \alpha \supset S. \\
\text{Case I: } \alpha = p_j \text{ (atomic). Since } \nu(p_j \supset S) = F, \text{ then } \nu(p_j) = T, \nu(S_j) = T \text{ for } i < n, \text{ and } \nu(S_n) = F. \\
\text{Subcase I-a: } j < n. \text{ Then } \nu(S_j) = \nu(p_j) = T, \text{ this forces } S_j = p_j \text{ and } S = S_1 \supset (S_2 \supset \ldots (S_{j-1} \supset S_j) \ldots). \text{ We have the following derivation of } p_j \supset S: \\
\end{align*}
\]

\[
\begin{align*}
& R1 \ p_j \supset S_n \\
& R1 \ S_{n-1} \supset (p_j \supset S_n) \\
& R7. \ p_j \supset (S_{n-1} \supset S_n) \\
& (R1 \& R7) 
\end{align*}
\]
\[ p_j \supset (S_{j+1} \supset \ldots (S_{n-1} \supset S_n) \ldots) \]

R2 \[ p_i \supset (p_i \supset (S_{i+1} \supset \ldots (S_{n-1} \supset S_n) \ldots) \]

R1 \[ S_{j-1} \supset (p_j \supset (p_{j+1} \supset \ldots (S_{n-1} \supset S_n) \ldots) \]

R7 \[ p_j \supset (S_{j-1} \supset (p_j \supset (S_{j+1} \supset \ldots (S_{n-1} \supset S_n) \ldots) \]

(R1 & R7) :

\[ \vdash p_j \supset S \]

**Subcase I-b: j = n.** Then \( v(S_n) = F \). Since \( v(p_n) = v(p_j) = T \) by the initial observation for Case I, we must have \( S_n = \sim p_n \). We have the deduction:

A1 \[ p_n \supset \sim p_n \]

R1 \[ S_{n-1} \supset (p_n \supset \sim p_n) \]

R7 \[ p_n \supset (S_{n-1} \supset \sim p_n) \]

(R1 & R7) :

\[ \vdash p_n \supset \sim p_n \]

**Case II: (inductive step) \( \alpha = \sim \beta \).**

**Subcase II-a: \( \beta = p_j \) with \( p_j \) atomic.** It is similar to Case I.

**Subcase II-b: \( \beta = \gamma \).** If \( v(\sim \gamma \supset S) = F \) then \( v(\gamma \supset S) = F \). By induction hypothesis: \( \vdash \sim \gamma \supset S \), by R6: \( \vdash \sim \sim \gamma \supset S \).

**Subcase II-c: \( \beta = (\gamma \supset \gamma') \).** If \( v(\sim (\gamma \supset \gamma') \supset S) = F \) then \( v(\gamma) = T \), \( v(\gamma') = F \), and \( v(S) = F \). Therefore, \( v(\gamma \supset S) = F \) and \( v(\sim \gamma \supset S) = F \). By induction hypothesis: \( \vdash \sim \gamma \supset S \) and \( \vdash \sim \sim \gamma \supset S \). By R8: \( \vdash \sim (\gamma \supset \gamma') \supset S \).

**Case III: (inductive step) \( \alpha = \gamma \supset \gamma' \).** If \( v((\gamma \supset \gamma') \supset S) = F \) then \( v(S) = F \), and \( v(\gamma) = F \) or \( v(\gamma') = T \). In the first case, \( v(\sim \gamma \supset S) = F \). By induction hypothesis: \( \vdash \sim \gamma \supset S \), and by R4: \( \vdash (\gamma \supset \gamma') \supset S \). In the second case, \( v(\gamma' \supset S) = F \). By inductive hypothesis: \( \vdash \sim \gamma' \supset S \), and by R3: \( \vdash (\gamma \supset \gamma') \supset S \).

To conclude the proof of the theorem, let \( v(\alpha) = F, P(\alpha) = \{ p_1, p_2, \ldots, p_n \} \) and define \( p_i'' = p_i \) if \( v(p_i) = T \), \( p_i'' = \sim p_i \) if \( v(p_i) = F \). Then \( v(p_i'') = T \) for \( i = 1, 2, \ldots, n \). Form the formula \( S = p_1'' \supset (p_2'' \supset \ldots (p_{n-1}'' \supset \sim p_n) \ldots) \).

We have \( v(S) = F \) and \( v(\sim \alpha \supset S) = F \). By property (*) above: \( \vdash \sim \alpha \supset S \), and by R5: \( \vdash \sim \alpha \). Q.E.D.

3 **Observations** If the propositional language contains the connective \( v \), it is enough to add the following rules to obtain completeness:

R9 (a) \[ \frac{\alpha \supset \beta}{(\alpha \lor \gamma) \supset \beta} \]

R9 (b) \[ \frac{\alpha \supset \beta}{(\gamma \lor \alpha) \supset \beta} \]
If the system contains the connective $\land$, the following rule will be enough to take care of it:

$$R10 \quad \frac{\alpha \supset (\beta \supset \gamma)}{(\alpha \land \beta) \supset \gamma}$$

Finally, it is not possible to give a similar deductive system for the non-valid formulae of the first-order predicate calculus because that would imply the decidability of the calculus.

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