Clifford theory for tensor categories

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Abstract
A graded tensor category over a group $G$ will be called a strongly $G$-graded tensor category if every homogeneous component has at least one multiplicatively invertible object. Our main result is a description of the module categories over a strongly $G$-graded tensor category as induced from module categories over tensor subcategories associated with the subgroups of $G$.

1. Introduction

Classical Clifford theory is an important collection of results relating the representation of a group to the representation of its normal subgroups. The principal results can be generalized using strongly graded rings, as in [7]. The goal of this paper is to describe a categorical analogue of the Clifford theory for tensor categories.

Throughout this article we work over a field $k$. By a tensor category $(\mathcal{C}, \otimes, \alpha, 1)$ we understand a $k$-linear abelian category $\mathcal{C}$, endowed with a $k$-bilinear exact bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, an object $1 \in \mathcal{C}$ and an associativity constraint $\alpha_{V,W,Z} : (V \otimes W) \otimes Z \rightarrow V \otimes (W \otimes Z)$ such that Mac Lane’s pentagon axiom holds [5], that is, $V \otimes 1 = 1 \otimes V = V$, $\alpha_{V,1,W} = \text{id}_{V \otimes W}$ for all $V, W \in \mathcal{C}$ and $\dim_k \text{End}_\mathcal{C}(1) = 1$.

An interesting and current problem is the classification of module categories over a tensor category (see [2, 12, 19–21]). A left module category over a tensor category $\mathcal{C}$, or a left $\mathcal{C}$-module category, is a $k$-linear abelian category $\mathcal{M}$ equipped with an exact bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ and natural isomorphisms $\alpha_{X,Y,M} : (X \otimes Y) \otimes M \rightarrow X \otimes (Y \otimes M)$, where $X, Y, Z \in \mathcal{C}$ and $M \in \mathcal{M}$, satisfying natural axioms.

Definition 1.1. Let $\mathcal{C}$ be a tensor category and let $\mathcal{M}$ be a $\mathcal{C}$-module category. A $\mathcal{C}$-submodule category of $\mathcal{M}$ is a Serre subcategory $\mathcal{N} \subseteq \mathcal{M}$ of $\mathcal{M}$ such that $\mathcal{N}$ is a $\mathcal{C}$-module category with respect to $\otimes$.

A $\mathcal{C}$-module category will be called simple if it does not contain any non-trivial $\mathcal{C}$-submodule category.

Remark 1.2. A rigid tensor category over an algebraically closed field is called finite if it is equivalent as an abelian category to the category of finite representation of a finite-dimensional algebra (see [13]). In this case the correct definition of a module category is that of an exact module category (see [13]). For exact module categories over finite tensor categories, the notion of a simple module category is equivalent to that of an indecomposable module category. In particular, a semisimple module category over a fusion category is simple if and only if it is indecomposable.
Let \( \mathcal{C} \) and \( \mathcal{D} \) be tensor categories. A \( \mathcal{C} \)-\( \mathcal{D} \)-bimodule category is a \( k \)-linear abelian category \( \mathcal{M} \), endowed with a structure of left \( \mathcal{C} \)-module category and right \( \mathcal{D} \)-module category such that the ‘actions’ commute up to natural isomorphisms in a coherent way. See Section 2 for details of the definitions of a \( \mathcal{C} \)-module category, a \( \mathcal{C} \)-bimodule category, a \( \mathcal{C} \)-module functor, a \( \mathcal{C} \)-linear natural transformation and their composition.

For a right \( \mathcal{C} \)-module category \( \mathcal{M} \) and a left \( \mathcal{C} \)-module category \( \mathcal{N} \), the tensor product category of \( k \)-linear module categories \( \mathcal{M} \boxtimes \mathcal{N} \) was defined in \([28]\); however, typically \( \mathcal{M} \boxtimes \mathcal{N} \) is not an abelian category. If \( \mathcal{M} \) is a \( \mathcal{D} \)-\( \mathcal{C} \)-bimodule category, then the category \( \mathcal{M} \boxtimes \mathcal{N} \) has a coherent left \( \mathcal{D} \)-action.

Let \( G \) be a group and let \( \mathcal{C} \) be a tensor category. We shall say that \( \mathcal{C} \) is \( G \)-graded if there is a decomposition

\[
\mathcal{C} = \bigoplus_{x \in G} \mathcal{C}_x
\]

of \( \mathcal{C} \) into a direct sum of full abelian subcategories such that, for all \( \sigma, x \in G \), the bifunctor \( \otimes \) maps \( \mathcal{C}_\sigma \times \mathcal{C}_x \to \mathcal{C}_{\sigma x} \) (see \([10]\)).

Recall that a graded ring \( A = \bigoplus_{x \in G} A_x \) is called strongly graded if \( A_x A_y = A_{xy} \) for all \( x, y \in G \). If we denote by \( \mathcal{C}_\sigma \cdot \mathcal{C}_\tau \subseteq \mathcal{C}_{\sigma \tau} \) the full \( k \)-linear subcategory of \( \mathcal{C}_{\sigma \tau} \) whose objects are direct sums of objects of the form \( V_\sigma \otimes W_\tau \) for \( V_\sigma \in \mathcal{C}_\sigma \), \( W_\tau \in \mathcal{C}_\tau \) and \( \sigma, \tau \in G \), then the definition of a strongly graded tensor category is the following.

**Definition 1.3.** Let \( \mathcal{C} = \bigoplus_{\sigma \in G} \mathcal{C}_\sigma \) be a graded tensor category over a group \( G \). We shall say that \( \mathcal{C} \) is strongly graded if the inclusion functor \( \mathcal{C}_\sigma \cdot \mathcal{C}_\tau \hookrightarrow \mathcal{C}_{\sigma \tau} \) is an equivalence of \( k \)-linear categories for all \( \sigma, \tau \in G \).

**Remark 1.4.** Note that \( \mathcal{C}_\sigma \cdot \mathcal{C}_\tau \) is only the full \( k \)-linear subcategory of \( \mathcal{C}_{\sigma \tau} \), and not the full abelian subcategory generated by \( \mathcal{C}_\sigma \cdot \mathcal{C}_\tau \). For example, the Tambara–Yamagami categories \( \text{TY}(A, \chi, \epsilon) \) (see \([29]\)) are \( \mathbb{Z}_2 \)-graded fusion categories that are not strongly graded. In fact, the simple objects of \( \mathcal{C}_0 \) are invertible and \( \mathcal{C}_1 \) only has one simple object. Then the objects of the category \( \mathcal{C}_1 \cdot \mathcal{C}_1 \) have the form \( (X \otimes X)^{\oplus n} \), and the full \( k \)-linear subcategory \( \mathcal{C}_1 \cdot \mathcal{C}_1 \) is not equivalent to \( \mathcal{C}_0 \) if \( \mathcal{C}_0 \) has more than one simple object. Note that the abelian subcategory of \( \mathcal{C}_0 \) generated by \( \mathcal{C}_1 \cdot \mathcal{C}_1 \) is equivalent to \( \mathcal{C}_0 \).

Also note that, for every tensor category \( (\mathcal{C}, \otimes, I) \), the \( k \)-linear category \( \mathcal{C} \cdot \mathcal{C} \) is equivalent to \( \mathcal{C} \) since \( V \cong I \otimes V \in \mathcal{C} \cdot \mathcal{C} \) for every \( V \in \text{Obj}(\mathcal{C}) \).

By Lemma 3.1 (see later), a graded tensor category over a group \( G \) is a strongly \( G \)-graded tensor category if and only if every homogeneous component has at least one invertible object. Let \( \mathcal{C} \) be a strongly \( G \)-graded tensor category. Given a \( \mathcal{C} \)-module category \( \mathcal{M} \), we shall denote by \( \Omega^{\mathcal{C} \cdot \mathcal{C}}(\mathcal{M}) \) the set of equivalences classes of simple \( \mathcal{C} \)-submodule categories of \( \mathcal{M} \). By Corollary 4.3 (see later), the group \( G \) acts on \( \Omega^{\mathcal{C} \cdot \mathcal{C}}(\mathcal{M}) \) by

\[
G \times \Omega^{\mathcal{C} \cdot \mathcal{C}}(\mathcal{M}) \longrightarrow \Omega^{\mathcal{C} \cdot \mathcal{C}}(\mathcal{M}), \quad (g, [X]) \longmapsto [C_g \boxtimes \mathcal{C} \cdot X].
\]

Our main result is the following.

**Theorem 1.5** (Clifford theorem for module categories). Let \( \mathcal{C} \) be a strongly \( G \)-graded tensor category and let \( \mathcal{M} \) be a simple abelian \( \mathcal{C} \)-module category. Then the following conditions hold.

(i) The action of \( G \) on \( \Omega^{\mathcal{C} \cdot \mathcal{C}}(\mathcal{M}) \) is transitive.
(ii) Let \( \mathcal{N} \) be a simple abelian \( \mathcal{C} \)-submodule subcategory of \( \mathcal{M} \). Let \( H = \text{st}(\mathcal{N}) \) be the stabilizer subgroup of \( \mathcal{N} \in \Omega_{\mathcal{C}}(\mathcal{M}) \), and also let

\[
\mathcal{M}_H = \sum_{h \in H} \mathcal{C}_h \boxtimes \mathcal{N}.
\]

Then \( \mathcal{M}_H \) is a simple \( \mathcal{C}_H \)-module category and \( \mathcal{M} \cong \mathcal{C} \boxtimes_{\mathcal{C}_H} \mathcal{M}_H \) as \( \mathcal{C} \)-module categories.

An important family of examples of strongly graded tensor categories is the crossed product tensor categories (see \([18, 27]\)). Let \( \mathcal{C} \) be a tensor category and let \( G \) be a group. We shall denote by \( G \) the monoidal category, where the objects are the elements of \( G \), arrows are identities and the tensor product is the product of \( G \).

Let \( \text{Aut}_\otimes(\mathcal{C}) \) be the monoidal category where objects are tensor auto-equivalences of \( \mathcal{C} \), arrows are tensor natural isomorphisms and the tensor product is the composition of functors. An action of the group \( G \) over a monoidal category \( \mathcal{C} \) is a monoidal functor \( * : G \to \text{Aut}_\otimes(\mathcal{C}) \).

Given an action \( * : G \to \text{Aut}_\otimes(\mathcal{C}) \) of \( G \) on \( \mathcal{C} \), the \( G \)-crossed product tensor category, denoted by \( \mathcal{C} \times G \), is defined as follows. As an abelian category \( \mathcal{C} \times G = \bigoplus_{\sigma \in G} \mathcal{C}_\sigma \), where \( \mathcal{C}_\sigma = \mathcal{C} \) as an abelian category, the tensor product is

\[
[X, \sigma] \otimes [Y, \tau] := [X \otimes \sigma_*(Y), \sigma \tau], \quad X, Y \in \mathcal{C}, \ \sigma, \tau \in G
\]

and the unit object is \([1, e] \). See \([27]\) for the associativity constraint and a proof of the pentagon identity.

The category \( \mathcal{C} \times G \) is \( G \)-graded by

\[
\mathcal{C} \times G = \bigoplus_{\sigma \in G} (\mathcal{C} \times G)_\sigma, \quad (\mathcal{C} \times G)_\sigma = \mathcal{C}_\sigma,
\]

and the objects \([1, \sigma] \in (\mathcal{C} \times G)_\sigma \) are invertible, with inverse \([1, \sigma^{-1}] \in (\mathcal{C} \times G)_{\sigma^{-1}} \).

Another useful construction of a tensor category starting from a \( G \)-action over a tensor category \( \mathcal{C} \) is the \( G \)-equivariantization of \( \mathcal{C} \), denoted by \( \mathcal{C}^G \). This construction has been used, for example, in \([3, 14, 17, 18, 27]\).

The category \( \mathcal{C} \) is a \( \mathcal{C} \times G \)-module category with action \([V, \sigma] \otimes W = V \otimes \sigma_*(W) \) (see \([18, 27]\)). Moreover, the tensor category of \( \mathcal{C} \times G \)-linear endofunctors of \( \mathcal{C} \), denoted by \( \mathcal{F}_{\mathcal{C} \times G}(\mathcal{C}, \mathcal{C}) \), is monoidally equivalent to the \( G \)-equivariantization \( \mathcal{C}^G \) of \( \mathcal{C} \) (see \([18]\)). With the help of this equivalence, we can describe the module categories over \( \mathcal{C}^G \) using the description of the module categories over the strongly \( G \)-graded tensor category \( \mathcal{C} \times G \) (see \([11]\) for the fusion category case).

The paper is organized as follows. Section 2 consists mainly of definitions and properties of module and bimodule categories over tensor categories and the tensor product of module categories, which will be needed in what follows. In Section 3 we introduce module categories graded over a \( G \)-set and give a structure theorem for them. In Section 4 the main theorem (Theorem 1.5) is proved. In Section 5 we describe the simple module categories over \( \mathcal{C} \times G \) and the simple module categories over \( \mathcal{C}^G \) if \( G \) is finite.

2. Preliminaries

A \( k \)-linear category or a category additive over \( k \) is a category in which the sets of arrows between two objects are \( k \)-vector spaces, the compositions are \( k \)-bilinear operations, finite direct sums exist and there is a zero object. A \( k \)-linear functor \( \mathcal{C} \to \mathcal{D} \) between \( k \)-linear categories is an additive functor that is \( k \)-linear on the spaces of morphisms. The notion of a \( k \)-bilinear bifunctor \( \mathcal{C} \times \mathcal{C}' \to \mathcal{D} \) is obvious.
DEFINITION 2.1 [19, Definition 6]. Let $\mathcal{C}$ be a monoidal category. A \textit{left $\mathcal{C}$-module category} over $\mathcal{C}$ is a category $\mathcal{M}$ together with a bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ and natural isomorphisms

$$m_{X,Y,M} : (X \otimes Y) \otimes M \rightarrow X \otimes (Y \otimes M)$$

such that

$$(\alpha_{X,Y,Z} \otimes M)m_{X,Y \otimes Z,M}(X \otimes m_{Y,Z,M}) = m_{X \otimes Y,Z,M}m_{X,Y,Z \otimes M},$$

$$1 \otimes M = M$$

for all $X, Y, Z \in \mathcal{C}$ and $M \in \mathcal{M}$.

A module category $\mathcal{M}$ over a tensor category $\mathcal{C}$ will always be abelian, and the bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ will always be exact. A right module category is defined in a similar way.

REMARK 2.2. For a category $\mathcal{M}$, the category of $k$-linear exact endofunctors $\mathcal{F}(\mathcal{M}, \mathcal{M})$ is a $k$-linear abelian strict monoidal category, where the kerner of morphism $\tau : F \rightarrow G$ in $\mathcal{F}(\mathcal{M}, \mathcal{M})$ is the functor $K : \mathcal{M} \rightarrow \mathcal{M}$, defined by $K(M) = \ker(\tau_M)$, and with the composition of functors as tensor product. For a tensor category $\mathcal{C}$, a structure of $\mathcal{C}$-module category $(\mathcal{M}, \otimes, m)$ on $\mathcal{M}$ is the same as an exact monoidal functor $(F, \zeta) : \mathcal{C} \rightarrow \mathcal{F}(\mathcal{M}, \mathcal{M})$. The bijection is given by the equation $V \otimes M = F(V)(M)$, identifying

$$(\zeta_{V,W})_M : (F(V) \circ F(W))(M) \rightarrow F(V \otimes W)(M)$$

with

$$m_{X,Y,M}^{-1} : V \otimes (W \otimes M) \rightarrow (V \otimes W) \otimes M.$$

EXAMPLE 2.3. Let $(A, m, e)$ be an associative algebra in $\mathcal{C}$. Let $\mathcal{C}_A$ be the category of right $A$-modules in $\mathcal{C}$. This is an abelian left $\mathcal{C}$-module category with action $V \otimes (M, \eta) = (V \otimes M, (id_V \otimes \eta)\alpha_{V,M,A})$ and associativity constraint $\alpha_{X,Y,M}$ for $X, Y \in \mathcal{C}$ and $M \in \mathcal{C}_A$ (see [19, Section 3.1]).

EXAMPLE 2.4. We shall denote by Vec$_f$ the category of finite-dimensional vector spaces over $k$. This is a semisimple tensor category with only one simple object. For every $k$-linear abelian category $\mathcal{M}$, there is a unique Vec$_f$-module category structure with action $k^\otimes n \otimes X := X^\otimes n$ (see [23, Lemma 2.2.2]).

EXAMPLE 2.5. Let $H$ be a Hopf algebra and let $B \subseteq A$ be a left faithfully flat $H$-Galois extension. Let $\mathcal{M}_B$ and $\mathcal{M}_H$ be the categories of right $B$-modules and right $H$-comodules, respectively. Recall that the category of right Hopf $(H, A)$-modules $\mathcal{M}_B^H$ is by definition the category $(\mathcal{M}_H)_A$ of right $A$-modules over $\mathcal{M}_H$. By Schneider’s structure theorem [26], the functor $\mathcal{M}_B \rightarrow (\mathcal{M}_H)_A$, $M \mapsto M \otimes_B A$, is a category equivalence with inverse $M \mapsto M^{coH}$. So $\mathcal{M}_B$ has an $\mathcal{M}_H$-module category structure as in Example 2.3.

For two $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$, a $\mathcal{C}$-linear functor or module functor $(F, \phi) : \mathcal{M} \rightarrow \mathcal{N}$ consists of an exact functor $F : \mathcal{M} \rightarrow \mathcal{N}$ and natural isomorphisms

$$\phi_{X,M} : F(X \otimes M) \rightarrow X \otimes F(M)$$

such that

$$(X \otimes \phi_{Y,M})\phi_{X,Y \otimes M}F(m_{X,Y,M}) = m_{X,Y,F(M)}\phi_{X \otimes Y,M}$$

for all $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$. 
If \( \mathcal{M} \) and \( \mathcal{N} \) are \( k \)-linear abelian categories, then \( \mathcal{F}_{\text{Vec}}(\mathcal{M}, \mathcal{N}) \) is the category of \( k \)-linear exact functors, and so \( \mathcal{F}_{\text{Vec}}(\mathcal{M}, \mathcal{N}) = \mathcal{F}(\mathcal{M}, \mathcal{N}) \).

A \( \mathcal{C} \)-linear natural transformation between \( \mathcal{C} \)-linear functors \((F, \phi), (F', \phi') : \mathcal{M} \rightarrow \mathcal{N}\) is a \( k \)-linear natural transformation \( \sigma : F \rightarrow F' \) such that

\[
\phi'_{X,M} \sigma_{X \otimes M} = (X \otimes \sigma_{M}) \phi_{X,M}
\]

for all \( X \in \mathcal{C} \) and \( M \in \mathcal{M} \).

We shall denote the category of \( \mathcal{C} \)-linear functors and \( \mathcal{C} \)-linear natural transformations between \( \mathcal{C} \)-module categories \( \mathcal{M} \) and \( \mathcal{N} \) by \( \mathcal{F}_\mathcal{C}(\mathcal{M}, \mathcal{N}) \).

**Definition 2.6.** Let \( \mathcal{C} \) be a tensor category and let \( \mathcal{M} \) be a \( \mathcal{C} \)-module category. A \( \mathcal{C} \)-submodule category of \( \mathcal{M} \) is a Serre subcategory \( \mathcal{N} \subseteq \mathcal{M} \) of \( \mathcal{M} \) such that it is a \( \mathcal{C} \)-module category with respect to \( \otimes \).

A \( \mathcal{C} \)-module category will be called **simple** if it does not contain any non-trivial \( \mathcal{C} \)-submodule category.

For \( \mathcal{C} \)-linear functors \((G, \psi) : \mathcal{D} \rightarrow \mathcal{M} \) and \((F, \phi) : \mathcal{M} \rightarrow \mathcal{N}\), their composition is a \( \mathcal{C} \)-linear functor \((F \circ G, \theta) : \mathcal{D} \rightarrow \mathcal{N}\), where

\[
\theta_{X,L} = \phi_{X,G(L)} F(\psi_{X,L})
\]

for \( X \in \mathcal{C} \) and \( L \in \mathcal{D} \). So we have a bifunctor

\[
\mathcal{F}_\mathcal{C}(\mathcal{M}, \mathcal{N}) \times \mathcal{F}_\mathcal{C}(\mathcal{D}, \mathcal{M}) \rightarrow \mathcal{F}_\mathcal{C}(\mathcal{D}, \mathcal{N}),
\]

\[
((F, \phi), (G, \psi)) \mapsto (F \circ G, \theta).
\]

**2.1. Strict module categories**

A monoidal category is called **strict** if its associativity constraint is the identity. In the same way, we say that a module category \((\mathcal{M}, \otimes, \alpha)\) over a strict monoidal category \((\mathcal{C}, \otimes, 1)\) is strict if \( \alpha \) is the identity.

The main result of this subsection establishes that every monoidal category \( \mathcal{C} \) is monoidally equivalent to a strict monoidal category \( \mathcal{C}' \) such that every module category over \( \mathcal{C}' \) is equivalent to a strict one.

**Lemma 2.7.** Let \( \mathcal{C} \) be a monoidal category. Then \( \mathcal{F}_\mathcal{C}(\mathcal{C}, \mathcal{C}) \cong \mathcal{C} \), where \( \mathcal{C} \) is a left \( \mathcal{C} \)-module category with the tensor product and the isomorphism of associativity. Moreover, \( \mathcal{C}^{\text{op}} \cong \mathcal{F}_\mathcal{C}(\mathcal{C}, \mathcal{C}) \) as monoidal categories (where \( \mathcal{C}^{\text{op}} = \mathcal{C} \) as categories, and the tensor product \( V \otimes^{\text{op}} W = W \otimes V \)).

**Proof.** We define the functor \((\_\_): \mathcal{C} \rightarrow \mathcal{F}_\mathcal{C}(\mathcal{C}, \mathcal{C})\) as follows: given \( V \in \mathcal{C} \), the functor \((\hat{V}, \alpha_{\_,-V}) : \mathcal{C} \rightarrow \mathcal{C}, W \mapsto W \otimes V \), \( \alpha_{X,Y,V} : \hat{V}(X \otimes Y) \rightarrow X \otimes \hat{V}(Y) \) is a \( \mathcal{C} \)-module functor. If \( \hat{\phi} : \hat{V} \rightarrow \hat{V}' \) is a morphism in \( \mathcal{C} \), then we define the natural transformation \( \hat{\phi} : \hat{V} \rightarrow \hat{V}' \) as \( \hat{\phi}_W = \text{id} \otimes \hat{\phi} : \hat{V}(W) = W \otimes V \rightarrow \hat{V}'(W) = W \otimes V' \).

The natural isomorphism

\[
\alpha_{\_,-W,V} : \hat{V} \circ \hat{W} \rightarrow V \otimes^{\text{op}} W
\]

gives a structure of monoidal functor to \((\_\_).\)

Let \((F, \psi) : \mathcal{C} \rightarrow \mathcal{C}\) be a module functor. Then we have a natural isomorphism

\[
\sigma_X = \psi_{X,1} : F(X) = F(X \otimes 1) \rightarrow X \otimes F(1) = \hat{F}(1)(X)
\]
such that
\[
\alpha_{X,Y,1} \sigma_{X} \otimes Y = \alpha_{X,Y,1}(1) \psi_{X} \otimes Y.
\]

that is, \(\sigma_{X}\) is a natural isomorphism module between \((F,\psi)\) and \((F(1),\alpha_{-,-,F(1)})\). So the functor is essentially surjective.

Let \(\phi: \hat{V} \to \hat{V}'\) be a \(C\)-linear natural morphism. Then \(\alpha_{X,1,V}\phi_{X} = \text{id}_{X} \otimes \phi_{1} \alpha_{X,1,V}'\), and so \(\phi_{X} = \text{id}_{X} \otimes \phi_{1}\), and the monoidal functor \((-)\) is faithful and full. Hence, by \[16\] Theorem 1, p. 91; \[22\], Proposition 4.4.2, the functor is an equivalence of monoidal categories.

**Proposition 2.8.** Let \(\mathcal{C}\) be a monoidal category. Then there is a strict monoidal category \(\overline{\mathcal{C}}\) such that every module category over \(\mathcal{C}\) is equivalent to a strict \(\mathcal{C}\)-module category and \(\overline{\mathcal{C}}\) is monoidally equivalent to \(\mathcal{C}\).

**Proof.** Let \(\overline{\mathcal{C}} = \mathcal{F}_{\mathcal{C}}(\mathcal{C},\mathcal{C})^{op}\). By Lemma 2.7, \(\overline{\mathcal{C}}\) is monoidally equivalent to \(\mathcal{C}\). Let \((\mathcal{M}, \otimes, m)\) be a left \(\mathcal{C}\)-module category. The category \(\mathcal{F}_{\mathcal{C}}(\mathcal{C},\mathcal{M})\) is a strict left \(\overline{\mathcal{C}}\)-module category with the composition of \(\mathcal{C}\)-module functors. Conversely, if \(\mathcal{M}'\) is a \(\overline{\mathcal{C}}\)-module category, then \(\mathcal{M}'\) is a module category over \(\mathcal{C}\), using the tensor equivalence \((-): \mathcal{C} \to \mathcal{F}_{\mathcal{C}}(\mathcal{C},\mathcal{C})\).

In a similar way to the proof of Lemma 2.7, the functor
\[
\mathcal{M} \to \mathcal{F}_{\mathcal{C}}(\mathcal{C},\mathcal{M}),
\]
\[
M \mapsto (\hat{M}, m_{-,-,M}),
\]
is an equivalence of \(\mathcal{C}\)-module categories. So every module category over \(\overline{\mathcal{C}}\) is equivalent to a strict one.

**2.2. Tensor product of module categories**

**Definition 2.9** \[28\], p. 518. Let \((\mathcal{M}, m)\) and \((\mathcal{N}, n)\) be right and left \(\mathcal{C}\)-module categories, respectively. A \(\mathcal{C}\)-bilinear functor \((F, \zeta): \mathcal{M} \times \mathcal{N} \to \mathcal{D}\) is a bifunctor \(F: \mathcal{M} \times \mathcal{N} \to \mathcal{D}\), together with natural isomorphisms
\[
\zeta_{M,X,N}: F(M \otimes X, N) \to F(X, M \otimes N),
\]
such that
\[
F(m_{M,X,Y,N})\zeta_{M,X,Y,N}F(n_{X,Y,N}) = \zeta_{M \otimes X,Y,N} \zeta_{M,X,Y \otimes N}
\]
for all \(M \in \mathcal{M}\), \(N \in \mathcal{N}\) and \(X, Y \in \mathcal{C}\).

A natural transformation \(\omega: (F, \zeta) \to (F', \zeta')\) between \(\mathcal{C}\)-bilinear functors is a natural transformation \(\omega_{M,N}: F(M, N) \to F'(M, N)\) such that
\[
\omega_{M,X \otimes N} \zeta_{M,X,N} = \alpha'_{M,X,N} \omega_{M \otimes X,N}
\]
for all \(M \in \mathcal{M}\), \(N \in \mathcal{N}\) and \(X \in \mathcal{C}\).

**Example 2.10.** Let \(\mathcal{C}\) be a tensor category and let \(\mathcal{D}\) be a tensor subcategory of \(\mathcal{C}\). Let \((\mathcal{M}, m)\) be a \(\mathcal{C}\)-module category and let \(\mathcal{N}\) be a \(\mathcal{D}\)-module subcategory of the \(\mathcal{D}\)-module category \(\mathcal{M}\). Then the functor \(\mathcal{C} \times \mathcal{N} \to \mathcal{M}\), \((V, N) \to V \otimes M\), has a canonical \(\mathcal{D}\)-bilinear structure. Here, \(\mathcal{C}\) is a \(\mathcal{D}\)-module category in the obvious way, and the \(\mathcal{D}\)-bilinear isomorphism is given by \(m\).
We shall denote by $\text{Bil}(\mathcal{M}, \mathcal{N}; D)$ the category of $C$-bilinear functors. In [28], a $k$-linear category (not necessarily abelian) $\mathcal{M} \boxtimes_C \mathcal{N}$ was constructed by generators and relations, together with a $C$-bilinear functor $T : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \boxtimes_C \mathcal{N}$, that induces an equivalence of $k$-linear categories $F(\mathcal{M} \boxtimes_C \mathcal{N}, D) \to \text{Bil}(\mathcal{M}, \mathcal{N}; D)$ for every $k$-linear category $D$.

The objects of $\mathcal{M} \boxtimes_C \mathcal{N}$ are finite sums of symbols $[X, Y]$ for objects $X \in \mathcal{M}$ and $Y \in \mathcal{N}$. Morphisms are sums of compositions of symbols

$$[f, g] : [X, Y] \to [X', Y']$$

for $f : X \to X'$ and $g : Y \to Y'$, symbols

$$\alpha_{X,V,Y} : [X \otimes V, Y] \to [X, V \otimes Y]$$

for $X \in \mathcal{M}$, $V \in C$ and $Y \in \mathcal{N}$, and symbols for the formal inverse of $\alpha_{X,V,Y}$. The generator morphisms satisfy the following relations.

1. Linearity:

$$[f + f', g] = [f, g] + [f', g], \quad [f, g + g'] = [f, g] + [f, g'],$$

for all morphisms $f, f' : M \to M'$ in $\mathcal{M}$, $g, g' : N \to N'$ in $\mathcal{N}$ and $a \in k$.

2. Functoriality:

$$[ff', gg'] = [f', g'][f, g], \quad [id_M, id_N] = id_{[M,N]}$$

for all $f : M \to M'$ and $f' : M' \to M''$ in $\mathcal{M}$, and $g : N \to N'$ and $g' : N' \to N''$ in $\mathcal{N}$.

3. Naturality:

$$\alpha_{M',V',N'}[f \otimes u, g] = [f, u \otimes g]\alpha_{M,V,N}$$

for morphisms $f : M \to M'$ in $\mathcal{M}$, $u : V \to V'$ in $C$ and $g : N \to N'$ in $\mathcal{N}$.

4. Coherence:

$$[\alpha_{M,V,W}, id_N]\alpha_{M,V \otimes W,N}[id_M, \alpha_{V,W,N}] = \alpha_{M \otimes V,Y,N}\alpha_{M,X,Y \otimes N},$$

for all $M \in \mathcal{M}$, $N \in \mathcal{N}$ and $V, W \in \mathcal{C}$. Let $\mathcal{M}$ and $\mathcal{N}$ be $k$-linear categories. Then the category $\mathcal{M} \boxtimes \mathcal{N} := \mathcal{M} \boxtimes_{\text{Vec}_F} \mathcal{N}$ is the tensor product of $k$-linear tensor categories (see [5, Definition 1.1.15]). If $\mathcal{M}$ and $\mathcal{N}$ are semisimple categories, then this is Deligne’s tensor product of abelian categories [9].

**Definition 2.11** [28, p. 517]. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be tensor categories. A $\mathcal{C}_1$-$\mathcal{C}_2$-bimodule category is a $k$-linear abelian category $\mathcal{M}$, equipped with exact bifunctors $\otimes : \mathcal{C}_1 \times \mathcal{M} \to \mathcal{M}$ and $\otimes : \mathcal{M} \times \mathcal{C}_2 \to \mathcal{M}$, and naturals isomorphisms

$$\alpha_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M),$$

$$\alpha_{X,M,S} : (X \otimes M) \otimes S \to X \otimes (M \otimes S),$$

$$\alpha_{M,S,T} : (M \otimes S) \otimes T \to M \otimes (S \otimes T)$$

for all $X, Y \in \mathcal{C}_1$, $M \in \mathcal{M}$ and $S, T \in \mathcal{C}_2$, such that $\mathcal{M}$ is a left $\mathcal{C}_1$-module category with $\alpha_{X,Y,M}$, it is a right $\mathcal{C}_2$-module category with $\alpha_{M,S,T}$, and

$$\text{id}_X \otimes \alpha_{Y,M,S} \alpha_{X,Y,M,S} \otimes \text{id}_S = \alpha_{X, Y, M \otimes Z} \alpha_{X, Y, M, S},$$

$$\text{id}_X \otimes \alpha_{M,T} \alpha_{X,M \otimes S} \alpha_{X,M,S} \otimes \text{id}_T = \alpha_{X, M \otimes S T} \alpha_{X, M, S T}.$$
The action over the morphisms $\alpha_{M,Y,N}$ is given by $\text{id}_X \otimes \alpha_{M,Y,N} = \alpha_{X \otimes M,X,Y,N} \circ [\alpha_{X,M,Y,N}]^{-1}$, and the associativity is

$$[\alpha_{X,Y,M},N] : [(X \otimes Y) \otimes M,N] \rightarrow [X \otimes (Y \otimes M),N].$$

**Proposition 2.12.** Let $\mathcal{C}$ be a tensor category. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be $\mathcal{C}$-bimodule categories and let $\mathcal{M}_3$ be a right $\mathcal{C}$-module category. Then the following hold:

(i) $\mathcal{C} \boxtimes \mathcal{C} \mathcal{M}_3 \cong \mathcal{M}_3$ as left $\mathcal{C}$-module categories;

(ii) $(\mathcal{M}_1 \boxtimes \mathcal{C} \mathcal{M}_2) \boxtimes \mathcal{C} \mathcal{M}_3 \cong \mathcal{M}_1 \boxtimes \mathcal{C} (\mathcal{M}_2 \boxtimes \mathcal{C} \mathcal{M}_3)$ as left $\mathcal{C}$-module categories;

(iii) if $\mathcal{M} = \bigoplus^n_i \mathcal{M}^i$ and $\mathcal{N} = \bigoplus^n_j \mathcal{N}^j$ as right and left $\mathcal{C}$-module categories, respectively, then $\mathcal{M} \boxtimes \mathcal{N} = \bigoplus_{i,j} \mathcal{M}_i \boxtimes \mathcal{N}_j$ as $k$-linear categories.

**Proof.** By Proposition 2.8, we can suppose that all module categories are strict.

(i) The functor $F : \mathcal{M} \rightarrow \mathcal{C} \boxtimes \mathcal{C} \mathcal{M}$, $M \mapsto [1,M]$ is a category equivalence. In effect, using the isomorphism $\alpha_{1,M,M}$, we can see that $F$ is essentially surjective, and every morphism between $[1,M]$ and $[1,N]$ is of the form $[1,f]$ for $f : M \rightarrow N$. Then $F$ is faithful and full. Moreover, with the natural isomorphism $\eta_{V,M} = \alpha_{1,V,M} : F(V \otimes N) \rightarrow V \otimes F(N)$, the pair $(F, \eta)$ is a $\mathcal{C}$-linear functor since

$$\eta_{V \otimes W,M} = \alpha_{1,V \otimes W,M} = \alpha_{V,W,M} \circ \alpha_{1,V,W \otimes M} = \text{id}_V \otimes \alpha_{1,V,W,M} \circ \eta_{V,W \otimes M} = \text{id}_V \otimes \eta_{W,M} \circ \eta_{V,W \otimes M}.$$

(ii) For every object $M_1 \in \mathcal{M}_1$, the functor $\lambda_{M_1} : \mathcal{M}_2 \times \mathcal{M}_3 \rightarrow (\mathcal{M}_1 \boxtimes \mathcal{C} \mathcal{M}_2) \boxtimes \mathcal{C} \mathcal{M}_3$, where

$$\lambda_{M_1}(M_2,M_3) = [[M_1,M_2],M_3], \quad \lambda_{M_1}(f,g) = [[\text{id}_{M_1},f],g],$$

with the natural transformation $\eta_{\lambda_{M_1},V,M_3} := \alpha_{[M_1,M_2],V,M_3}$, is a $\mathcal{C}$-bilinear functor. So we have a family of functors $\lambda_{M_1} : \mathcal{M}_2 \boxtimes \mathcal{C} \mathcal{M}_3 \rightarrow (\mathcal{M}_1 \boxtimes \mathcal{C} \mathcal{M}_2) \boxtimes \mathcal{C} \mathcal{M}_3$, $\lambda_{M_1}([[M_2,M_3]]) = [[M_1,M_2],M_3]$. Now, the functor

$$(\mathcal{M}_1 \times (\mathcal{M}_2 \boxtimes \mathcal{C} \mathcal{M}_3)) \rightarrow (\mathcal{M}_1 \boxtimes \mathcal{C} \mathcal{M}_2) \boxtimes \mathcal{C} \mathcal{M}_3,$$

$$(M_1,[M_2,M_3]) \mapsto \lambda_{M_1}([[M_2,M_3]]),$$

with the natural transformation $\eta_{\lambda_{M_1},V,[M_2,M_3]} = \alpha_{M_1,V,[M_2,M_3]}$, is a $\mathcal{C}$-bilinear functor. So we have a functor $\pi : \mathcal{M}_1 \boxtimes \mathcal{C} (\mathcal{M}_2 \boxtimes \mathcal{C} \mathcal{M}_3) \rightarrow (\mathcal{M}_1 \boxtimes \mathcal{C} \mathcal{M}_2) \boxtimes \mathcal{C} \mathcal{M}_3$, $[M_1,[M_2,M_3]] \mapsto [[M_1,M_2],M_3]$. The functor $\pi$ is essentially surjective and

$$\pi([f,[g,h]]) = [[f,g],h],$$

$$\pi([\text{id}_{M_1},\alpha_{M_2,V,M_3}]) = \alpha_{[M_1,M_3],V,M_3},$$

$$\pi(\alpha_{M_1,V,[M_2,M_3]}) = [\alpha_{M_1,V,M_2},\text{id}_{M_3}].$$

So $\pi$ is faithful and full, and hence, by [16, Theorem 1, p. 91], the functor $\pi$ is a category equivalence. Finally, note that the functor $\pi$ is $\mathcal{C}$-linear.

(iii) This result follows directly by the construction of $\mathcal{M} \boxtimes \mathcal{C} \mathcal{N}$. □

Let $\mathcal{C}$ be a $G$-graded tensor category. Note that, if $H \subseteq G$ is a subgroup of $G$, then the category $\mathcal{C}_H = \bigoplus_{\tau \in H} \mathcal{C}_{\tau}$ is a tensor subcategory of $\mathcal{C}$.

We shall say that an object $U \in \mathcal{C}$ is invertible if the functor $U \otimes (-) : \mathcal{C} \rightarrow \mathcal{C}, V \mapsto U \otimes V$ is a category equivalence or, equivalently, if there is an object $U^* \in \mathcal{C}$ such that $U^* \otimes U \cong U \otimes U^* \cong 1$. 

Proposition 2.13. Let $\mathcal{C}$ be a $G$-graded category and let $H \subseteq G$ be a subgroup of $G$. Suppose that every category $\mathcal{C}_\sigma$ has at least one invertible object for every $\sigma \in G$. Let $\mathcal{M}$ be a module category over $\mathcal{C}_H = \bigoplus_{h \in H} \mathcal{C}_h$. Then the $k$-linear category $\mathcal{C} \boxtimes_{\mathcal{C}_H} \mathcal{M}$ is an abelian category. Moreover, since $\mathcal{C}$ is a $\mathcal{C} \boxtimes_{\mathcal{C}_H} \mathcal{M}$ is a left module category over $\mathcal{C}_H$.

Proof. We shall suppose that the tensor category $\mathcal{C}$ is strict. Let $\Sigma = \{e, \sigma_1, \ldots\}$ be a set of representatives of the cosets $G/H$. Since $\mathcal{C} = \bigoplus_{\sigma \in \Sigma} \mathcal{C}_{\sigma H}$ as right $\mathcal{C}_{\sigma H}$-module categories, $\mathcal{C} \boxtimes_{\mathcal{C}_H} \mathcal{M} = \bigoplus_{\sigma \in \Sigma} \mathcal{C}_{\sigma H} \boxtimes_{\mathcal{C}_H} \mathcal{M}$ as $k$-linear categories, by Proposition 2.12.

For every coset $\sigma H$ in $G$, let $U_\sigma \in \mathcal{C}_\sigma$ be an invertible object. The functor $U_\sigma : \mathcal{C}_H \to \mathcal{C}_{\sigma H}$, $V \mapsto U_\sigma \otimes V$ is a category equivalence with a quasi-inverse $U^*_\sigma : \mathcal{C}_{\sigma H} \to \mathcal{C}_H$, $W \mapsto U^*_\sigma \otimes W$. Then we can assume, up to isomorphisms, that every object of $\mathcal{C}_{\sigma H}$ is of the form $U_\sigma \otimes V$, where $V \in \mathcal{C}_H$.

Let $\bigoplus [V_\sigma, M_\sigma] \in \mathcal{C}_{\sigma H} \boxtimes_{\mathcal{C}_H} \mathcal{M}$. For every $V_\sigma$, there exist $V'_\sigma$ such that $V_\sigma = U_\sigma \otimes V'_\sigma$. Then $\bigoplus [V_\sigma, M_\sigma] \cong [U_\sigma, \bigoplus V'_\sigma \otimes M_\sigma]$, that is, we can assume, up to isomorphisms, that every object of $\mathcal{C}_{\sigma H} \boxtimes_{\mathcal{C}_H} \mathcal{M}$ is of the form $[U_\sigma, M]$. If $U_\sigma \otimes V \cong U_\tau \otimes V$, then $V \cong 1$; so every morphism $[U_\sigma, M] \to [U_\tau, M']$ is of the form $[id_{U_\sigma}, f]$, where $f : M \to M'$. Then the functor $\mathcal{M} \to \mathcal{C}_{\sigma H} \boxtimes_{\mathcal{C}_H} \mathcal{M}, f \mapsto [id_{U_\sigma}, f]$ is an equivalence of $k$-linear categories. We define the abelian structure over $\mathcal{C}_{\sigma H} \boxtimes_{\mathcal{C}_H} \mathcal{M}$ as the structure induced by this equivalence.

For the second part, note that $\mathcal{C} \boxtimes_{\mathcal{C}_H} \mathcal{M} = \bigoplus_{\sigma \in \Sigma} \mathcal{C}_{\sigma H} \boxtimes_{\mathcal{C}_H} \mathcal{M}$ as an abelian category. So we need to prove that, if

$$0 \longrightarrow [U_\sigma, S] \longrightarrow [U_\sigma, T] \longrightarrow [U_\sigma, W] \longrightarrow 0 \quad (2.1)$$

is an exact sequence in $\mathcal{C}_{\sigma H} \boxtimes_{\mathcal{C}_H} \mathcal{M}$, then the sequence

$$0 \longrightarrow [X \otimes U_\sigma, S] \longrightarrow [X \otimes U_\sigma, T] \longrightarrow [X \otimes U_\sigma, W] \longrightarrow 0 \quad (2.2)$$

is exact for all $X \in \mathcal{C}$. Since $\mathcal{C} = \bigoplus_{\sigma \in \Sigma} \mathcal{C}_\sigma$, we can suppose that $X \in \mathcal{C}_e$, and then $[X \otimes U_\sigma, S], [X \otimes U_\sigma, T], [X \otimes U_\sigma, W] \in \mathcal{C}_{\tau \sigma H} \boxtimes_{\mathcal{C}_H} \mathcal{M}$.

Let $U_{\tau \sigma} \in \mathcal{C}_{\tau \sigma}$ with inverse object $U^*_{\tau \sigma} \in \mathcal{C}_{(\tau \sigma)^{-1}}$. So we have the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & [X U_\sigma, S] & \stackrel{[id, f]}{\longrightarrow} & [X U_\sigma, T] & \stackrel{[id, g]}{\longrightarrow} & [X U_\sigma, W] & \longrightarrow & 0 \\
\downarrow & & \downarrow & \stackrel{[id, f]}{\longrightarrow} & \downarrow & \stackrel{[id, g]}{\longrightarrow} & \downarrow & \\
0 & \longrightarrow & [U_{\tau \sigma} (U^*_{\tau \sigma} X U_\sigma), S] & \stackrel{[id, f]}{\longrightarrow} & [U_{\tau \sigma} (U^*_{\tau \sigma} X U_\sigma), T] & \stackrel{[id, g]}{\longrightarrow} & [U_{\tau \sigma} (U^*_{\tau \sigma} X U_\sigma), W] & \longrightarrow & 0 \\
\downarrow & \alpha_{U_{\tau \sigma} U^*_{\tau \sigma} X U_\sigma, S} & \alpha_{U_{\tau \sigma} U^*_{\tau \sigma} X U_\sigma, T} & \alpha_{U_{\tau \sigma} U^*_{\tau \sigma} X U_\sigma, W} & \downarrow & \downarrow & \downarrow & \\
0 & \longrightarrow & [U_{\tau \sigma}, (U^*_{\tau \sigma} X U_\sigma) S] & \stackrel{[id, id_{U^*_{\tau \sigma} X U_\sigma}] f}{\longrightarrow} & [U_{\tau \sigma}, (U^*_{\tau \sigma} X U_\sigma) T] & \stackrel{[id, id_{U^*_{\tau \sigma} X U_\sigma}] g}{\longrightarrow} & [U_{\tau \sigma}, (U^*_{\tau \sigma} X U_\sigma) W] & \longrightarrow & 0, \\
\end{array}
$$

where tensor symbols between objects and morphisms have been omitted as a space-saving measure.
Then the sequence (2.2) is exact if and only if the sequence
\[ 0 \rightarrow [U_{\tau_\sigma}, (U_{\tau_\sigma}^* XU_{\sigma})S] \rightarrow [U_{\tau_\sigma}, (U_{\tau_\sigma}^* XU_{\sigma})T] \rightarrow [U_{\tau_\sigma}, (U_{\tau_\sigma}^* XU_{\sigma})W] \rightarrow 0 \] (2.3)
is exact. By definition, the sequence (2.3) is exact if and only if the sequence
\[ 0 \rightarrow (U_{\tau_\sigma}^* XU_{\sigma})S \rightarrow (U_{\tau_\sigma}^* XU_{\sigma})T \rightarrow (U_{\tau_\sigma}^* XU_{\sigma})W \rightarrow 0 \]
in \( \mathcal{M} \) is exact. However, since \( \mathcal{M} \) is a \( \mathcal{C}_H \)-module category, it is exact. \( \square \)

3. Strongly graded tensor categories

Recall from Definition 1.3 that the \( G \)-graded category \( \mathcal{C} \) is called strongly graded if the inclusion functor \( \mathcal{C}_\sigma \cdot \mathcal{C}_\tau \rightarrow \mathcal{C}_{\sigma \tau} \) is a category equivalence for all \( \sigma, \tau \in G \).

**Lemma 3.1.** Let \( \mathcal{C} \) be a tensor category. Then \( \mathcal{C} \) is strongly graded over \( G \) if and only if the category \( \mathcal{C}_\sigma \) has at least one multiplicatively invertible element for all \( \sigma \in G \). Moreover, in this case the Grothendieck ring of \( \mathcal{C} \) is a \( G \)-crossed product.

**Proof.** Since \( \mathcal{C} \) is strongly graded there exist objects \( V_1, \ldots, V_n \in \mathcal{C}_\sigma \) and \( W_1, \ldots, W_t \in \mathcal{C}_{\sigma^{-1}} \) such that \( 1 \cong \bigoplus_{i,j} V_i \otimes W_j \), then \( \text{End}_G(\bigoplus_{i,j} V_i \otimes W_j) \cong \text{End}_G(1) \cong k \), and so \( n = 1 \) and \( t = 1 \), that is, there exist objects \( V \in \mathcal{C}_\sigma \) and \( W \in \mathcal{C}_{\sigma^{-1}} \) such that \( V \otimes W \cong 1 \).

Conversely, suppose that \( \mathcal{C}_\sigma \) has at least one invertible object for all \( \sigma \in G \). Let \( U_{\sigma} \in \mathcal{C}_\sigma \) be an invertible object with dual object \( U_{\sigma}^* \in \mathcal{C}_{\sigma^{-1}} \), and so \( V \cong U_{\sigma} \otimes (U_{\sigma}^* \otimes V) \) for every \( V \in \mathcal{C}_{\sigma \tau} \). Then the inclusion functor is essentially surjective, and therefore it is an equivalence.

Recall that, by definition, a graded ring \( A = \bigoplus_{\sigma \in G} A_\sigma \) is a crossed product over \( G \) if for all \( \sigma \in G \) the abelian group \( A_\sigma \) has at least one invertible element. Thus, by the first part of the lemma, the Grothendieck ring of \( \mathcal{C} \) is a \( G \)-crossed product if \( \mathcal{C} \) is strongly graded. \( \square \)

**Example 3.2.** Let \( \text{Vec}_G^\omega \) be the semisimple category of finite-dimensional \( G \)-graded vector spaces, with constraint of associativity \( \omega(a, b, c)\text{id}_{abc} \) for all \( a, b, c \in G \), where \( \omega \in Z^3(G, k^*) \) is a 3-cocycle. Then \( \text{Vec}_G^\omega \) is a strongly \( G \)-graded tensor category.

**Example 3.3.** Let \( \mathcal{C} \times G \) be a crossed product tensor category. By Section 1, the category \( \mathcal{C} \times G \) is a strong \( G \)-graded tensor category. Now, if we take a normalized 3-cocycle \( \beta \in Z^3(G, k^*) \) and we define a new associator \( \alpha_{\beta}^{[U,\sigma],[V,\tau],[W,\rho]} = \beta(\sigma,\tau,\rho)\alpha_{[U,\sigma],[V,\tau],[W,\rho]} \), then the new tensor category is also strongly \( G \)-graded.

3.1. Module categories graded over a \( G \)-set

**Definition 3.4.** Let \( \mathcal{C} = \bigoplus_{\sigma \in G} \mathcal{C}_\sigma \) be a graded tensor category and let \( X \) be a left \( G \)-set. A \( \text{left } X \text{-graded } \mathcal{C} \text{-module category} \) is a \( \mathcal{C} \)-module category \( \mathcal{M} \) endowed with a decomposition
\[ \mathcal{M} = \bigoplus_{x \in X} \mathcal{M}_x \]
into a direct sum of full abelian subcategories such that, for all \( \sigma \in G \) and \( x \in X \), the bifunctor \( \otimes \) maps \( \mathcal{C}_\sigma \times \mathcal{M}_x \) to \( \mathcal{M}_{\sigma x} \).

An \( X \)-graded \( \mathcal{C} \)-module functor \( F : \mathcal{M} \rightarrow \mathcal{N} \) is a \( \mathcal{C} \)-module functor such that \( F(\mathcal{M}_x) \) is mapped to \( \mathcal{N}_x \) for all \( x \in X \).
Definition 3.5. A left \(X\)-graded \(C\)-submodule category of \(M\) is a Serre subcategory \(N\) of \(M\) such that \(N\) is an \(X\)-graded \(C\)-module category with respect to \(\otimes\), and the grading \(N_x \subseteq M_x\), where \(x \in X\).

An \(X\)-graded \(C\)-module category will be called simple if it contains no non-trivial \(X\)-graded \(C\)-submodule category.

Lemma 3.6. Let \(C\) be a \(G\)-graded tensor category and let \(H \subseteq G\) be a subgroup of \(G\). If \(N\) is a left \(C_H\)-module category, then the category \(C \otimes_{C_H} N\) is a \(G/H\)-graded \(C\)-module category with grading \((C \otimes_{C_H} N)_{\sigma H} = (\bigoplus_{\tau \in \sigma H} C_\tau) \otimes_{C_H} N\).

Proof. Let \(\Sigma = \{e, \sigma_1, \ldots\}\) be a set of representatives of the cosets of \(G\) modulo \(H\). By Proposition 2.12, \(C \otimes_{C_H} N = \bigoplus_{\sigma \in \Sigma} C_{\sigma H} \otimes_{C_H} N\) as \(k\)-linear categories, and, by the definition of the action of \(C\), the module category \(C \otimes_{C_H} N\) is \(G/H\)-graded.

Proposition 3.7. Let \(C\) be a strongly \(G\)-graded tensor category and let \((A, m, e)\) be an algebra in \(C_H\). Then \(C \otimes_{C_H} (C_H)_A \cong C_A\) as \(G/H\)-graded \(C\)-module categories.

Proof. Let \(\Sigma = \{e, \sigma_1, \ldots\}\) be a set of representatives of the cosets of \(G\) modulo \(H\). The \(C\)-module category \(C_A\) has a canonical \(G/H\)-grading, that is, if \((M, \rho)\) is an \(A\)-module, then

\[(M, \rho) = \bigoplus_{\sigma \in \Sigma} (M_{\sigma H}, \rho_{\sigma H}),\]

where \(M_{\sigma H} = \bigoplus_{h \in H} M_{\sigma h}\) and \(\rho_{\sigma H} = \bigoplus_{h \in H} \rho_{\sigma h}\).

Let us consider the canonical \(C\)-linear functor \(F : C \otimes_{C_H} (C_H)_A \to C_A\),

\[[V, (M, \rho)] \mapsto (V \otimes M, \text{id}_V \otimes \rho).\]

We shall first show that \(F\) is a category equivalence.

Let \(U_\sigma \in C_{\sigma H}\) be an invertible object for every coset of \(H\) on \(G\). Let \((M, \rho) \in C_A\) be a homogeneous \(A\)-module of degree \(\sigma^{-1} H\). Then the \(A\)-module \((U_\sigma \otimes M, \text{id}_{U_\sigma} \otimes \rho)\) is also an \(A\)-module in \(C_H\) and \(F([U_{\sigma^{-1}}, (U_\sigma \otimes M, \text{id}_{U_\sigma} \otimes \rho)]) \cong (M, \rho) \in C_A\). So \(F\) is an essentially surjective functor.

We can suppose, up to isomorphisms, that every object of \(C_{\sigma H} \otimes_{C_H} (C_H)_A\) is of the form \([U_\sigma, (M, \rho)]\). Then \(F([U_\sigma, (M, \rho)]) = (U_{\sigma^{-1}} \otimes M, \text{id}_{U_{\sigma^{-1}}} \otimes \rho)\). Now it is clear that the functor \(F\) is faithful and full, and so, by [16, Theorem 1, p. 91], the functor \(F\) is a category equivalence.

Theorem 3.8. Let \(C\) be a strongly graded tensor category over a group \(G\) and let \(X\) be a transitive \(G\)-set. Let \(M\) and \(N\) be non-zero \(X\)-graded module categories. Then the following hold:

(i) \(M \cong C \otimes_{C_H} M_x\) as \(X\)-graded \(C\)-module categories, where, for all \(x \in X\), we have that \(H = \text{st}(x)\) is the stabilizer subgroup of \(x \in X\);

(ii) there is a bijective correspondence between isomorphism classes of \(X\)-graded \(C\)-module functors \((F, \eta) : M \to N\) and \(C_H\)-module functors \((T, \rho) : M_x \to N_x\).

Proof. (i) Choose \(x \in X\), and define \(H = \text{st}(x)\). In a similar way to the proof of Proposition 3.7, the canonical functor \(\mu : C \otimes_{C_H} M_x \to M\),

\[[V, M] \mapsto V \otimes M,\]

is a category equivalence and it respects the grading.
The proof of part (i) of the theorem is completed by showing that the functor $\mu$ is a $C$-module functor. Indeed, by Proposition 2.8, we can assume that the module categories are strict, and hence

$$\mu(V \otimes [W, M_x]) = \mu([V \otimes W, M_x]) = (V \otimes W) \otimes M_x = V \otimes (W \otimes M_x) = V \otimes \mu([W, M_x]),$$

that is, $\mu$ is a $C$-module functor.

(ii) By part (i), we can suppose that $N = C \boxtimes_H N_x$. Let $(F, \mu) : N_x \to M_x$ be a $C_H$-module functor. The functor $I(F) : C \times N_x \longrightarrow M$

$$(S, N) \mapsto S \otimes F(N)$$

with the natural transformation $id_S \otimes \mu_{V,N} : I(F)(S, V \otimes N) \to I(S \otimes V, N)$ is a $C_H$-bilinear functor. So we have a functor $I(F) : C \boxtimes_C H N_x \longrightarrow M$

$[S, N] \mapsto S \otimes F(N)$

and this is an $X$-graded $C$-module functor in the obvious way.

Let $(F = \bigoplus_{s \in X} F_s, \eta) : C \boxtimes_{C_H} N_x \to M$ be an $X$-graded $C$-module functor. Consider the natural isomorphism

$$\sigma_{[V,N]} := \eta_{V,[1,N]} : F([V, N]) \longrightarrow V \otimes F_x([1, N]) = I(F_x)([V, N]),$$

$$\sigma_{X \otimes [V,N]} = \eta_{X \otimes V,[1,N]} = id_X \otimes \eta_{V,[1,N]} \circ \eta_{X,[V,N]} = id_X \otimes \sigma_{[V,N]} \circ \eta_{X,[V,N]}.$$

So $\sigma$ is a natural isomorphism of module functors. 

COROLLARY 3.9. Let $C$ be a strongly $G$-graded tensor category. Then there is a bijective correspondence between module categories over $C_e$ and $G$-graded $C$-module categories.

This corollary follows as a particular case of Theorem 3.8 with $X = G$.

PROPOSITION 3.10. For every $\sigma, \tau \in G$, the canonical functor

$$f_{\sigma,\tau} : C_\sigma \boxtimes_{C_e} C_\tau \longrightarrow C_{\sigma \tau}, \quad f_{\sigma,\tau}([X,Y]) = X \otimes Y,$$

is an equivalence of $C_e$-bimodule categories.

Proof. Let us consider the graded $C$-module category $C(\tau)$, where $C = C(\tau)$ as $C$-module categories, but with the grading $(C(\tau))_g = C_{\tau g}$, for $\tau \in G$.

Since $C(\tau)_e = C_\tau$, by Theorem 3.8, the canonical functor $\mu(C(\tau)_e) : C \boxtimes_{C_e} C_\tau \to C(\tau)$,

$$[X,Y] \mapsto X \otimes Y$$

is an equivalence of $G$-graded $C$-module categories. So the restriction $\mu(C(\tau))_\sigma : C_\sigma \boxtimes_{C_e} C_\tau \to C(\tau)_\sigma = C_{\tau \sigma}$ is a $C_e$-module category equivalence. However, by definition, $\mu(C(\tau))_\sigma = f_{\sigma,\tau}$. It is clear that $f_{\sigma,\tau}$ is a $C_e$-bimodule category functor, and so the proof is finished.
4. Clifford theory

In this section we shall suppose that \( C \) is a strongly graded tensor category over a group \( G \).

We shall denote by \( \Omega_C \) the set of equivalences classes of simple \( C \)-module categories. Given a \( C \)-module category \( \mathcal{M} \), we shall denote by \( \Omega_C(\mathcal{M}) \) the set of equivalences classes of simple \( C \)-submodule categories of \( \mathcal{M} \).

**Lemma 4.1.** Let \( \mathcal{M} \) be a \( C \)-module category. Then, for all \( \sigma \in G \), the category \( \mathcal{C}_\sigma \ltimes \mathcal{C}_e \mathcal{M} \) is a simple \( C \)-module category if and only if \( \mathcal{M} \) is a simple \( C \)-module category.

**Proof.** If \( \mathcal{N} \) is a proper \( C \)-submodule category of \( \mathcal{M} \), then the category \( \mathcal{C}_\sigma \ltimes \mathcal{C}_e \mathcal{N} \) is a \( C \)-module category of \( \mathcal{C}_\sigma \ltimes \mathcal{C}_e \mathcal{M} \), and so the \( C \)-module category \( \mathcal{C}_\sigma \ltimes \mathcal{C}_e \mathcal{M} \) is not simple.

By Proposition 3.10, we have that \( \mathcal{M} \cong \mathcal{C}_g^{-1} \ltimes \mathcal{C}_e \mathcal{C}_g \ltimes \mathcal{C}_e \mathcal{M} \). So, if \( \mathcal{C}_g \ltimes \mathcal{C}_e \mathcal{M} \) is not simple, then \( \mathcal{M} \) is not simple either.

By Lemma 4.1 and Proposition 3.10, the group \( G \) acts on \( \Omega_C \) by \( G \times \Omega_C \rightarrow \Omega_C, (g, [X]) 
\rightarrow [\mathcal{C}_g \ltimes \mathcal{C}_e \mathcal{X}] \).

**Proposition 4.2.** Let \( \mathcal{M} \) be a \( C \)-module category and let \( \mathcal{N} \) be a \( C \)-submodule category of \( \mathcal{M} \). Then \( \mathcal{C}_\sigma \ltimes \mathcal{C}_e \mathcal{N} \cong \mathcal{C}_\sigma \otimes \mathcal{N} \) as \( C \)-module categories for all \( \sigma \in G \).

**Proof.** Define a \( G \)-graded \( C \)-module category by \( \text{gr-} \mathcal{N} = \bigoplus_{\sigma \in G} \mathcal{C}_\sigma \boxtimes \mathcal{N} \), with action \([V, T] = V_\sigma \otimes T\).

Since \( \mathcal{C}_e \boxtimes \mathcal{N} = \mathcal{N} \) as a \( C_e \)-module category, by Theorem 3.8, the canonical functor \( \mu(\mathcal{N}) : \mathcal{C}_e \boxtimes \mathcal{N} \rightarrow \text{gr-} \mathcal{N} \) is a category equivalence of \( G \)-graded \( C \)-module categories and the restriction \( \mu_\sigma : \mathcal{C}_\sigma \boxtimes \mathcal{N} \rightarrow \mathcal{C}_\sigma \boxtimes \mathcal{N} \) is a \( C_e \)-module category equivalence.

**Corollary 4.3.** Let \( \mathcal{M} \) be a \( C \)-module category. The action of \( G \) on \( \Omega_C \) induces an action of \( G \) on \( \Omega_C(\mathcal{M}) \).

**Proof.** Let \( \mathcal{N} \) be a simple \( C \)-submodule category of \( \mathcal{M} \). By Proposition 4.2, the functor \( \mu_\sigma : \mathcal{C}_\sigma \boxtimes \mathcal{N} \rightarrow \mathcal{C}_\sigma \boxtimes \mathcal{N} \) is a \( C_e \)-module category equivalence, and so \( \mathcal{C}_\sigma \boxtimes \mathcal{N} \) is equivalent to a \( C_e \)-submodule category of \( \mathcal{M} \).

Let \( \mathcal{M} \) be an abelian category and let \( \mathcal{N} \) and \( \mathcal{N}' \) be Serre subcategories of \( \mathcal{M} \). We shall denote by \( \mathcal{N} + \mathcal{N}' \) the Serre subcategory of \( \mathcal{M} \), where \( \text{Ob}(\mathcal{N} + \mathcal{N}') = \{ \text{subquotients of } N \oplus N' : N \in \mathcal{N}, N' \in \mathcal{N}' \} \). It will be called the sum category of \( \mathcal{N} \) and \( \mathcal{N}' \).
Proof of Theorem 1.5. (i) Let \( \mathcal{N} \) be a simple abelian \( \mathcal{C}_e \)-submodule category of \( \mathcal{M} \). The canonical functor
\[
\mu: \mathcal{C} \boxtimes_{\mathcal{C}_e} \mathcal{N} \rightarrow \mathcal{M},
\]
\[
[V, N] \mapsto V \otimes N,
\]
is a \( \mathcal{C} \)-module functor and \( \mu = \bigoplus_{\sigma \in G} \mu_\sigma \), where \( \mu_\sigma = \mu|_{\mathcal{C}_e} \). By Proposition 4.2, each \( \mu_\sigma \) is a \( \mathcal{C}_e \)-module category equivalence with \( \mathcal{C}_e \boxtimes N \).

Since \( \mathcal{M} \) is simple, every object \( M \in \mathcal{M} \) is isomorphic to some subquotient of \( \mu(X) \) for some object \( X \in \mathcal{C} \boxtimes_{\mathcal{C}_e} \mathcal{N} \). Then \( \mathcal{M} = \bigoplus_{\sigma \in G} \mathcal{C}_e \boxtimes N \) and each \( \mathcal{C}_e \boxtimes N \) is an abelian simple \( \mathcal{C}_e \)-submodule category.

Let \( S \) and \( S' \) be simple abelian \( \mathcal{C}_e \)-submodule categories of \( \mathcal{M} \). Then there exist \( \sigma, \tau \in G \) such that \( C_{\sigma} \boxtimes_{\mathcal{C}_e} \mathcal{N} \cong S \), \( C_{\tau} \boxtimes_{\mathcal{C}_e} \mathcal{N} \cong S' \), and, by Proposition 3.10, \( S' \cong C_{\tau^{-1}} \boxtimes_{\mathcal{C}_e} S \). So the action is transitive.

(ii) Let \( H = \text{st}(\{\mathcal{N}\}) \) be the stabilizer subgroup of \( \{\mathcal{N}\} \in \Omega_{\mathcal{C}_e}(\mathcal{M}) \) and let
\[
\mathcal{M}_N = \bigoplus_{h \in H} \mathcal{C}_e \boxtimes_{\mathcal{C}_e} \mathcal{N}.
\]

Since \( H \) acts transitively on \( \Omega_{\mathcal{C}_e}(\mathcal{M}_N) \), the \( \mathcal{C}_H \)-module category \( \mathcal{M}_N \) is simple. Let \( \Sigma = \{e, \sigma_1, \ldots\} \) be a set of representatives of the cosets of \( G \) modulo \( H \). The map \( \phi: G/H \rightarrow \Omega_{\mathcal{C}_H}(\mathcal{M}), \phi(\sigma H) = [C_{\sigma} \boxtimes_{\mathcal{C}_e} \mathcal{M}_N] \) is an isomorphism of \( G \)-sets. Then \( \mathcal{M} \) has a structure of \( G/H \)-graded \( \mathcal{C} \)-module category, where \( \mathcal{M} = \bigoplus_{\sigma \in \Sigma} C_{\sigma} \boxtimes_{\mathcal{C}_e} \mathcal{M}_N \). By Theorem 3.8, \( \mathcal{M} \cong \mathcal{C} \boxtimes_{\mathcal{C}_e} \mathcal{M}_N \) as \( \mathcal{C} \)-module categories. \( \square \)

Remark 4.4. Gelaki and Nikshych noted the existence of a grading by a transitive \( G \)-set for every indecomposable module category over a \( G \)-graded fusion category [15, Proposition 5.1]. Using Theorem 3.8 and [15, Proposition 5.1], we can obtain an alternative proof of the main theorem (Theorem 1.5) in the case of strongly graded fusion categories.
Thus $C^G$ has a tensor product defined by

$$(V, f) \otimes (W, g) := (V \otimes W, h),$$

where

$$h_{\sigma} = u_{\sigma} u_{\sigma} \psi(\sigma)^{-1}_{V,W},$$

and unit object $(1, \text{id}_1)$.

**Example 5.1** (The comodule category of a cocentral cleft exact sequence of Hopf algebras). Let $G$ be a group and let

$$k \longrightarrow A \longrightarrow H \longrightarrow kG \longrightarrow k$$

be a cocentral cleft exact sequence of Hopf algebras, that is, the projection $\pi : H \to kG$ admits a $kG$-colinear section $j : kG \to H$ that is invertible with respect to convolution product.

Since the sequence is cleft, the Hopf algebra $H$ has the structure of a bicrossed product $H \cong A^\tau \#_\sigma kG$ with respect to a certain compatible datum $(\cdot, \rho, \sigma, \tau)$, where $\cdot : A \otimes kG \to A$ is a weak action, $\sigma : kG \otimes kG \to A$ is an invertible cocycle, $\rho : kG \to kG \otimes A$ is a weak coaction and $\tau : kG \to A \otimes A$ is a dual cocycle, subject to compatibility conditions in [1, Theorem 2.20].

The projection in (5.2) is called cocentral if $\pi(h_1) \otimes h_2 = \pi(h_2) \otimes h_1$. This is equivalent to the weak coaction $\rho$ being trivial (see [17, Lemma 3.3]).

**Lemma 5.2.** Let $H \cong A^\tau \#_\sigma kG$ be a bicrossed product with a trivial coaction. Then the group $G$ acts over the category of right $A$-modules $A^M$, and $H^M \cong (A^M)^G$ as tensor categories, where $H^M$ is the category of right $H$-modules.

For the proof of Lemma 5.2, see [17, Lemma 3.3].

**Remark 5.3.** Let $H$ be a semisimple Hopf algebra over $C$. By [15, Proof of Theorem 3.8], the fusion category $H^M$ of finite-dimensional comodules is $G$-graded (not necessary strongly graded) if and only if there is a cocentral exact sequence of Hopf algebras as in (5.2). In this case, the fusion category $H^M$ is weakly Morita equivalent to a $G$-crossed tensor category $A^M \rtimes G$, that is, $H^M \cong F_{A^M \rtimes G}(N, N)$ for some indecomposable $A^M \rtimes G$-module category $N$.

5.2. The obstruction to a $G$-action over a tensor category

Let $C$ be a tensor category. We shall denote by $\text{Aut}_{\otimes}(C)$ the group of tensor auto-equivalences; it is the set of isomorphism classes of auto-equivalences of $C$, with multiplication induced by the composition, that is, $[F][F'] = [F \circ F']$.

Every $G$-action over a tensor category induces a group homomorphism $\psi : G \to \text{Aut}_{\otimes}(C)$. We shall say that a homomorphism $\psi : G \to \text{Aut}_{\otimes}(C)$ is realizable if there exists some $G$-action such that the induced group homomorphism coincides with $\psi$.

The goal of this subsection is show that, for every homomorphism $\psi : G \to \text{Aut}_{\otimes}(C)$, there exists an associated element in a third cohomology group that is zero if and only if $\psi$ is realizable. Moreover, every realization is in correspondence (not natural) with an element of a second cohomology group.

5.2.1. Categorical-groups. A categorical-group $G$ is a monoidal category where every object and every arrow are invertible. We refer the reader to [4] for a detailed exposition on the subject.
A trivial example of a categorical-group is the discrete categorical-group $\mathcal{G}$, which is associated to a group $G$. The objects of $\mathcal{G}$ are the elements of $G$, the arrows are simply the identities and the tensor product is the multiplication of $G$.

The complete invariants of a categorical-group $\mathcal{G}$ with respect to monoidal equivalences are

$$\pi_0(\mathcal{G}), \pi_1(\mathcal{G}), \alpha \in H^3(\pi_0(\mathcal{G}), \pi_1(\mathcal{G})),$$

where $\pi_0(\mathcal{G})$ is the group of isomorphism classes of objects, and $\pi_1(\mathcal{G})$ is the abelian group of automorphisms of the unit object. The group $\pi_1(\mathcal{G})$ is a $\pi_0(\mathcal{G})$-module in the natural way, and $\alpha$ is a third cohomology class given by the associator.

The complete invariants of a monoidal functor $F : \mathcal{G} \rightarrow \mathcal{G}'$ between categorical-groups with respect to monoidal isomorphisms are

$$\pi_0(F) : \pi_0(\mathcal{G}) \longrightarrow \pi_0(\mathcal{G}'), \quad \pi_1(F) : \pi_1(\mathcal{G}) \longrightarrow \pi_1(\mathcal{G}'), \quad \theta(F) : \pi_0(\mathcal{G}) \times \pi_0(\mathcal{G}) \longrightarrow \pi_1(\mathcal{G}'),$$

where $\pi_0(F)$ is a morphism of groups, $\pi_1(F)$ is a morphism of $\pi_0(\mathcal{G})$-modules and $\theta(F)$ is a class in $C^2(\pi_0(\mathcal{G}), \pi_1(\mathcal{G}'))/B^2(\pi_0(\mathcal{G}), \pi_1(\mathcal{G}'))$, such that

$$\delta(\theta(F)) = \pi_1(\mathcal{G}')\pi_0(\mathcal{G}) - \pi_0(\mathcal{G}')\pi_0(\mathcal{G}),$$

where

$$\pi_0(F)^* : C^*(\pi_0(\mathcal{G}'), \pi_1(\mathcal{G}')) \longrightarrow C^*(\pi_0(\mathcal{G}), \pi_1(\mathcal{G}')),$$

are the maps of cochain complexes induced by the group morphisms $\pi_0(F)$ and $\pi_1(F)$, respectively. The next result follows from the last discussion, or see [4].

**Proposition 5.4.** Let $\mathcal{G}$ be a categorical group and let $f : G \rightarrow \pi_0(\mathcal{G})$ be a morphism of groups. Then there is a monoidal functor $F : \mathcal{G} \rightarrow \mathcal{G}$ such that $f = \pi_0(F)$ if and only if the cohomology class of $f_\ast(\phi)$ is zero.

If $f_\ast(\phi)$ is zero, then the classes of equivalence of monoidal functors $F : \mathcal{G} \rightarrow \mathcal{G}$ are in one-to-one correspondence with $H^2(G, \pi_1(\mathcal{G})).$

**Proof.** The monoidal category $\mathcal{G}$ has invariants $\pi_0(\mathcal{G}) = G$ and $\pi_1(\mathcal{G}) = 0$. Then the proof follows from the discussions of this subsection, or see [4].

5.2.2. The obstruction to a $G$-action over a tensor category and cyclic actions. Let $\text{Aut}_\otimes(\mathcal{C})$ be the monoidal category of tensor auto-equivalences of a tensor category $\mathcal{C}$, where the arrows are tensor natural isomorphisms and the tensor product is given by a composition of functors. Then $\text{Aut}_\otimes(\mathcal{C})$ is a categorical-group.

The invariants associated to $\text{Aut}_\otimes(\mathcal{C})$ (see Section 5.2.1) are then $\pi_0(\text{Aut}_\otimes(\mathcal{C})) = \text{Aut}_\otimes(\mathcal{C})$ and $\pi_1(\text{Aut}_\otimes(\mathcal{C})) = \text{Aut}_\otimes(\text{id}_\mathcal{C})$, the group of monoidal natural isomorphisms of the identity functor.

**Theorem 5.5.** Let $\mathcal{C}$ be a tensor category and let $G$ be a group. Consider the data $(\text{Aut}_\otimes(\mathcal{C}), \text{Aut}_\otimes(\text{id}_\mathcal{C}), [a])$ associated to the categorical-group $\text{Aut}_\otimes(\mathcal{C})$. Then the following conditions hold

(i) A group homomorphism $f : G \rightarrow \text{Aut}_\otimes(\mathcal{C})$ is realized as a $G$-action over $\mathcal{C}$ if and only if $0 = [f_\ast(a)] \in H^3(G, \text{Aut}_\otimes(\text{id}_\mathcal{C})).$

(ii) If the group homomorphism $f : G \rightarrow \text{Aut}_\otimes(\mathcal{C})$ is realizable, then the set of realizations of $f$ is in one-to-one correspondence with $Z^2(G, \text{Aut}_\otimes(\text{id}_\mathcal{C}))$, and the set of equivalences classes of realizations of $f$ is in one-to-one correspondence with $H^2(G, \text{Aut}_\otimes(\text{id}_\mathcal{C})).$
Theorem 5.5 follows as is a particular case of Proposition 5.4.
Recall that, if $A$ is a module for the cyclic group $C_m$ of order $m$, then
\[
H^n(C_m; A) = \begin{cases} 
\{ a \in A : Na = 0 \} / (\sigma - 1)A & \text{if } n = 1, 3, 5, \ldots, \\
A^{C_m} / NA & \text{if } n = 2, 4, 6, \ldots,
\end{cases}
\] (5.3)
where $N = 1 + \sigma + \sigma^2 + \ldots + \sigma^{m-1}$ (see [31, Theorem 6.2.2]). Given an element $a \in A^{C_m}$, the associated 2-cocycle can be constructed as follows:
\[
\gamma_a(\sigma^i, \sigma^j) = \begin{cases} 
1 & \text{if } i + j < m, \\
a^{i+j-m} & \text{if } i + j \geq m.
\end{cases}
\] (5.4)

Let $F : \mathcal{C} \to \mathcal{C}$ be a monoidal equivalence such that there exists a monoidal natural isomorphism $\alpha : F^m \to \text{id}_\mathcal{C}$. By Theorems 5.5 and (5.3), the induced homomorphism $\psi : C_m \to \text{Aut}_{\otimes}(\mathcal{C})$ is realizable if and only if $\text{id}_F \otimes \alpha \otimes \text{id}_{F^{-1}} = \alpha$. In this case, two natural isomorphisms $\alpha_1, \alpha_2 : F^m \to \text{id}_\mathcal{C}$ realize equivalent $C_m$-actions if and only if there exists a monoidal natural isomorphism $\theta : F_1 \to F_2$ such that $\theta^m F_1 = F_2$.

**Corollary 5.6.** Let $\mathcal{C}$ be a tensor category and let $C_m$ be a cyclic group of order $m$. Then the set of $C_m$-actions over $\mathcal{C}$ is in one-to-one correspondence with pairs $(F, \alpha)$, where $F : \mathcal{C} \to \mathcal{C}$ is a monoidal equivalence, and $\alpha : F^m \to \text{id}_\mathcal{C}$ is a monoidal natural isomorphism such that $\text{id}_F \otimes \alpha = \alpha \otimes \text{id}_F$.

Two pairs $(F_1, \alpha_1)$ and $(F_2, \alpha_2)$ induce equivalent $C_m$-actions if and only if there exists a monoidal natural isomorphism $\theta : F_1 \to F_2$ such that $\theta^m F_1 = F_2$.

The description of the 2-cocycle associated to a $C_m$-invariant element (5.4) is as follows: the $C_m$-action $\psi : C^m \to \text{Aut}_{\otimes}(\mathcal{C})$ associated to a pair $(F, \alpha)$ is given by $\psi(1) = \text{id}_\mathcal{C}$ and $\psi(\sigma^i) = F^i$, where $i = 1, \ldots, m - 1$, and the monoidal natural isomorphisms $\phi_\alpha(\sigma^i, \sigma^j) : F^i \circ F^j \to F^{i+j}$ are given by
\[
\phi_\alpha(\sigma^i, \sigma^j) = \begin{cases} 
\text{id}_\mathcal{C} & \text{if } i + j < m, \\
\text{id}_F \otimes \alpha^{i+j-m} = \alpha^{i+j-m} \otimes \text{id}_F & \text{if } i + j \geq m.
\end{cases}
\] (5.5)

**5.2.3. The bigalois group of a Hopf algebra.** Let $H$ be a Hopf algebra. A right $H$-Galois object is a non-zero right $H$-comodule algebra $A$ such that the linear map defined by can : $A \otimes A \to A \otimes H, a \otimes b \mapsto ab(0) \otimes b(1)$, is bijective.

A fibre functor $F : H.\mathcal{M} \to \text{Vec}_k$ is an exact and faithful monoidal functor that commutes with colimits. Ulrich defined in [30] a fibre functor $F_A$ associated with each $H$-Galois object $A$, in the form $F_A(V) = A \square_H V$, where $A \square_H V$ is the cotensor product over $H$ of the right $H$-comodule $A$ and the left $H$-comodule $V$. He showed in [30] that this defines a category equivalence between $H$-Galois objects and fibre functors over $H.\mathcal{M}$.

Similarly, a left $H$-Galois object is a non-zero left $H$-comodule algebra $A$ such that the linear map can : $A \otimes A \to H \otimes A, a \otimes b \mapsto a(-1) \otimes a(0)b$ is bijective.

Let $H$ and $Q$ be Hopf algebras. An $H$-$Q$-bigalois object is an algebra $A$ that is an $H$-$Q$-bicomodule algebra and both a left $H$-Galois object and a right $Q$-Galois object.

Let $A$ be an $H$-Galois object. Schauenburg showed in [24, Theorem 3.5] that there is a Hopf algebra $L(A, H)$ such that $A$ is an $L(A, H)$-$H$-bigalois object.

The Hopf algebra $L(A, H)$ is the Tannakian–Krein reconstruction from the fibre functor associated to $A$. By [24, Corollary 5.7], the following categories are equivalent:

1. the monoidal category $\text{BiGal}(H)$, where objects are $H$-bigalois objects, morphisms are morphisms of $A$-bicomodule algebras, and the tensor product $A \square_H B$ is the cotensor product over $H$.
Schauenburg defined the group \( \text{BiGal}(H) \) as the set of isomorphism classes of \( H \)-bigalois objects with multiplication induced by the cotensor product. This group coincides with \( \text{Aut}_\otimes(H,M) \).

It is easy to see that, for the Hopf algebra \( kG \) of a group \( G \), we have \( \text{BiGal}(kG) = \text{Aut}(G) \rtimes H^2(G,k^*) \). However, it is difficult to find an explicit description in general. The group \( \text{BiGal}(H) \) has been calculated for some Hopf algebras, for example, Taft algebras \cite{25}, monomial non-semisimple Hopf algebras \cite{6}, and the algebra of functions over a finite group that are coprime to 6 (see \cite{8}).

5.2.4. The abelian group \( \text{Aut}_\otimes(\text{id}_C) \) for Hopf algebras. We have the following proposition.

**Proposition 5.7.** Let \( H \) be a Hopf algebra. Then \( \text{Aut}_\otimes(\text{id}_{H,M}) \cong G(H) \cap Z(H) \), which is the group of central group-likes of \( H \).

**Proof.** The maps \( H \otimes_k (M \otimes_k N) \to (H \otimes_H N) \otimes_k (H \otimes_H N) \), \( h \otimes m \otimes n \mapsto (h(1) \otimes m) \otimes (h(2) \otimes n) \) and \( H \otimes_k k \to k, h \otimes 1 \mapsto \epsilon(h) \) induce natural \( H \)-module morphisms

\[
F_{M,N} : H \otimes_H (M \otimes N) \to (H \otimes_H M) \otimes (H \otimes_H N),
\]

\[
F^0 : H \otimes_H k \to k.
\]

The identity monoidal functor is naturally isomorphic to \((H, \otimes_H (-), F,F^0)\), and it is well known that every \( H \)-bimodule endomorphism is of the form \( \psi_c : H \to H, h \mapsto ch \) for some \( c \in Z(H) \). The natural transformation associated to \( \psi_c \) is monoidal if and only if \( \psi_c \) is a bimodule coalgebra map, that is, if \( c \) is a group-like. \( \Box \)

For the group algebra \( kG \), we have \( \text{Aut}_\otimes(\text{id}_{kG,M}) \cong Z(G) \), which is the centre of \( G \), and for a Hopf algebra \( \mathbb{C}^G \), where \( G \) is a finite group, we have \( \text{Aut}_\otimes(\text{id}_{\mathbb{C}^G,M}) \cong G/[G,G] \).

Let \( \mathcal{C} \) be a complex fusion category, that is, a semisimple tensor category with finitely many isomorphism classes of simple objects. In \cite{15} it was shown that every fusion category is naturally graded by a group \( U(\mathcal{C}) \) called the universal grading group of \( \mathcal{C} \). The group \( U(\mathcal{C}) \) only depends on the Grothendieck ring of \( \mathcal{C} \).

In \cite[Proposition 3.9]{15} it was shown that, if \( \mathcal{C} \) is a fusion category and \( G = U(\mathcal{C}) \) is the universal grading group of \( \mathcal{C} \), then \( \text{Aut}_\otimes(\text{id}_{\mathcal{C}}) \cong \hat{G}_{ab} \), which is the group of characters of the maximal abelian quotient of \( G \).

**Corollary 5.8.** Let \( H \) be a semisimple almost-commutative Hopf algebra. Then \( U(H,M) \cong Z(H) \cap G(H) \).

**Proof.** Since \( H \) is almost-cocommutative, the Grothendieck ring is commutative, and hence the universal grading group is abelian. By Proposition 5.7 and \cite[Proposition 3.9]{15}, \( U(H,M) \cong Z(H) \cap G(H) \). \( \Box \)

5.3. \( G \)-invariant actions on module categories

If \( (F,\xi) : \mathcal{C} \to \mathcal{C} \) is a tensor functor and \( (\mathcal{M},\otimes,m) \) is a module category over \( \mathcal{C} \) we shall denote by \( \mathcal{M}^F \) the module category \( (\mathcal{M},\otimes^F,m^F) \) over the same underlying abelian category with action and associativity isomorphisms defined by

\[
X \otimes^F M = F(X) \otimes M, \quad m^F_{X,Y,M} = m_{F(X),F(Y),M}(\xi_{X,Y}^{-1} \otimes \text{id}_M),
\]

for all \( X,Y \in \mathcal{C}, M \in \mathcal{M} \).
Definition 5.9. Let \( C \) be a tensor category, \( M \) be a left \( C \)-module category and \( \sigma : C \to C \) be a monoidal functor. We shall say that the functor \( (T, \eta) : M \to M^\sigma \) is a \( \sigma \)-equivariant functor of \( M \) if it is a \( C \)-module functor.

Given an action of a group \( G \) over \( C \), the module category \( M \) is called \( G \)-invariant if there exists a \( \sigma \)-equivariant functor for each \( \sigma \in G \).

If \( \sigma, \tau \in G \), and \( T \) is \( \sigma \)-invariant and \( U \) is \( \tau \)-invariant then \( T \circ U \) is \( (\sigma \tau) \)-invariant. Indeed, let us assume that the functors \( (T, c) : M \to M^\sigma, (U, d) : M \to M^\tau \), are module functors then \( (T \circ U, b) : M \to M^{\sigma \tau} \) is a module functor, where

\[
b_{X,M} = (\gamma_{\sigma,\tau} X \otimes \text{id}) c_{\tau, X} M T(d_{X,M}), \tag{5.6}
\]

for all \( X \in C, M \in M \).

Given a \( G \)-action over a monoidal category \( C \) and a \( G \)-invariant module category \( M \), we denote by \( \text{Aut}_G^C(M) \) the following monoidal category: objects are \( \sigma \)-equivariant functors for all \( \sigma \in G \), morphisms are natural isomorphisms of module functors, the tensor product is the composition defined in 5.6 and the unit object is the identity functor of \( M \).

Definition 5.10. Let \( (\sigma, \phi(\sigma, \tau), \psi(\sigma)) : G \to \text{Aut}_G(C) \) be an action of \( G \) over a tensor category \( C \), and let \( M \) be a \( G \)-invariant \( C \)-module category. A \( G \)-invariant functor over \( M \) is a monoidal functor \( (\sigma^*, \phi, \psi) : G \to \text{Aut}_G^C(M) \) such that \( \sigma^* \) is a \( \sigma \)-invariant functor for each \( \sigma \in G \).

Remark 5.11. (i) A \( C \)-module category \( M \) with a \( G \)-invariant functor was called a \( G \)-equivariant \( C \)-module category in [11, definition 5.2].

(ii) Let \( C \) be a \( G \)-invariant monoidal category. The monoidal category \( \text{Aut}_G^C(M) \) is a graded categorical-group and the group \( \text{Aut}_G^C(M) \) has a natural group epimorphism \( \pi : \text{Aut}_G^C(M) \to G \). So, if a group homomorphism \( \psi : G \to \text{Aut}_G^C(M) \) is realizable, then \( \pi \psi = \text{id}_G \). Such group homomorphisms will be called split.

(iii) Let \( \psi : G \to \text{Aut}_G^C(M) \) be a split group homomorphism. If \( a \in H^3(\text{Aut}_G^C(M), H) \) is the 3-cocycle associated to the categorical-group \( \text{Aut}_G^C(M) \), then, as in Theorem 5.5, \( \psi \) is realizable if and only if the 3-cocycle \( \psi_*(a) \) is a 3-coboundary, and the set of realizations of \( \psi \) is in correspondence with the elements of a second cohomology group.

The following result appears in [27, Section 2].

Proposition 5.12. Let \( C \rtimes G \) be a crossed product tensor category. Then there is a bijective correspondence between structures of \( C \rtimes G \)-module category and \( G \)-invariant functors over a \( C \)-module category \( M \).

Proof. Let \( M \) be a \( C \rtimes G \)-module category. Each object \([1, \sigma] \), where \( \sigma \in G \), defines an equivalence \( \sigma : M \to M, M \mapsto [1, \sigma] \otimes M \). With the constraint of associativity \( \phi(\sigma, \tau)_M = \alpha_{([1, \tau], [1, \sigma])} \), this defines a monoidal functor \( G \to \text{Aut}(M) \).

The category \( M \) is a \( C \)-module category with \( V \otimes M = [V, e] \otimes V \) and, since \([1, \sigma] \otimes [V, e] = [\sigma(V), e] \otimes [1, \sigma]\), we have a natural isomorphism \( \psi(\sigma) : \sigma(V) \otimes \sigma(M) \to \sigma(V \otimes M) \), given by \( \psi(\sigma)V,M : \sigma(V) \otimes \sigma(M) \to \sigma(V \otimes M), \) this defines a \( G \)-invariant functor.

Conversely, if \( G \to \text{Aut}_G^C(M) \) is a \( G \)-invariant functor, then we have natural isomorphisms \( \phi(\sigma, \tau)_M : \sigma \tau_* (M) \to \sigma \tau_* (M) \) and \( \psi(\sigma)_V,M : \sigma(V) \otimes \sigma(M) \to \sigma(V \otimes M) \). Then we may define
the action on $\mathcal{M}$ by
\[(V, \sigma) \otimes M := V \otimes \sigma_s(M),\]
and constraint of associativity
\[\alpha(V, \sigma), (W, \tau) \otimes M = \text{id}_V \otimes \sigma_s(W) \otimes \phi(\sigma, \tau)_M \circ \alpha_{V, \sigma_s(W), \sigma_s(M)} \circ \text{id}_V \otimes \psi(\sigma)_W^1 \otimes M.\]

Suppose that the group $G$ is finite and that the tensor category $\mathcal{C}$ is a fusion category over an algebraically closed field of characteristic zero. Then the module categories over $G$ are in bijective correspondence by \cite{18, Proposition 3.2}. If $\mathcal{M}$ is $\mathcal{C} \rtimes G$-module category, then, by Proposition 5.12, there is a $G$-action on $\mathcal{M}$, and the category $\mathcal{M}^G$ is a $\mathcal{C}^G$-module category with
\[(V, f) \otimes (M, g) := (V \otimes M, h),\]
where
\[h_\sigma = g_\sigma h_\sigma \psi(\sigma)^{-1}_V \otimes M.\]

For a $k$-linear monoidal category and $G$ finite, where $\text{char}(k) \nmid |G|$, Theorem 4.1 of \cite{27} says that every $\mathcal{C}^G$-module category is of the form $\mathcal{M}^G$ for a $\mathcal{C} \rtimes G$-module category. The following result for fusion categories and finite groups appears in \cite{11}.

**Theorem 5.13.** Simple module categories over $\mathcal{C} \rtimes G$ are in bijective correspondence with the following data:

(i) a subgroup $H \subseteq G$;

(ii) a simple $H$-invariant $\mathcal{C}$ module category $\mathcal{M}$;

(iii) a monoidal functor $H \rightarrow \text{Aut}_\mathcal{C}^H(\mathcal{M})$.

If the group $G$ is finite, then the module categories over $\mathcal{C}^G$ are in bijection with the same data.

**Proof.** By Theorem 1.5, if $\mathcal{N}$ is a simple $\mathcal{C} \rtimes G$-module category, then it is isomorphic to $\mathcal{C} \rtimes H \mathcal{M}$ for some subgroup $H \subseteq G$ and a simple $\mathcal{C} \rtimes H$-module category $\mathcal{M}$ such that $\mathcal{M}$ is $H$-invariant. In particular, it follows that the restriction of $\mathcal{M}$ to $\mathcal{C}$ is simple. The correspondence now follows from Proposition 5.12.

If the group $G$ is finite, then the correspondence follows from \cite[Theorem 4.1]{27} or \cite[Proposition 3.2]{18}.

Suppose that $G$ is a finite group and $H \cong \mathbb{A}^r \#_{\sigma} kG$ is a bicrossed product with a trivial coaction. Then the module categories over $H \mathcal{M}$ are of the form $N^G$ for some $G$-equivariant $A\mathcal{M}$-module category $\mathcal{N}$. Moreover, the module category is simple if and only if $\mathcal{N}$ is simple.

**Example 5.14.** Let $N \geq 2$ be an integer and let $q \in \mathbb{C}$ be a primitive $N$th root of unity. The *Taft algebra* $T(q)$ is the $\mathbb{C}$-algebra presented by the generators $g$ and $x$ with the relations $g^N = 1$, $x^N = 0$ and $gx = qxg$. The algebra $T(q)$ carries a Hopf algebra structure, determined by
\[\Delta g = g \otimes g, \quad \Delta x = x \otimes 1 + g \otimes x.\]

Then $\varepsilon(g) = 1$, $\varepsilon(x) = 0$, $S(g) = g^{-1}$ and $S(x) = -g^{-1}x$. The following statements are known:

(i) $T(q)$ is a pointed non-semisimple Hopf algebra;

(ii) the group of group-like elements of $T(q)$ is $G(T(q)) = \langle g \rangle \cong \mathbb{Z}/(N)$;

(iii) $T(q) \cong T(q^*)$;

(iv) $T(q) \cong T(q')$ if and only if $q = q'$.
Proposition 5.15. Let $G$ be a group. Then the set of $G$-actions on the tensor category $T(q)^{\mathcal{M}}$ of $T(q)$-comodules is in one-to-one correspondence with the set of group homomorphisms from $G$ to $\mathbb{C}^* \ltimes \mathbb{C}$, where $\mathbb{C}^*$ acts on $\mathbb{C}$ by $\mathbb{C}^* \times \mathbb{C} \to \mathbb{C}$, $(s, t) \mapsto st$.

Proof. By Proposition 5.7, the abelian group $\text{Aut}_{\mathcal{M}}(id_{\mathcal{C}})$ is trivial, and, by [25, Theorem 5], $\text{Aut}_{\mathcal{M}}(T(q)^{\mathcal{M}}) = \text{BiGal}(T(q)) \cong \mathbb{C}^* \times \mathbb{C}$. Then, by Theorem 5.5, the set of isomorphism classes of $G$-actions is given by the set of group homomorphisms from $G$ to $\mathbb{C}^* \times \mathbb{C}$. $\square$

If $G = \mathbb{Z}/(N)$, then, by Proposition 5.15, the possible $G$-actions are parameterized by pairs $(r, \mu)$, where $r$ is a non-trivial $N$th root of unity and $\mu \in \mathbb{C}$.

We shall denote by $A_{(r, \mu)}$ the $T(q)$-bigalois object associated to the pair $(r, \mu) \in \mathbb{C}^* \times \mathbb{C} \cong \text{BiGal}(T(q))$ (see [25, Theorem 5]).

The $T(q)^{\mathcal{M}}$-module categories of rank 1 are in correspondence with fibre functors on $T(q)^{\mathcal{M}}$, and these are in turn in one-to-one correspondence with $T(q)$-Galois objects. By [25, Theorem 2], every $T(q)$-Galois object is isomorphic to $A_{(1, \beta)}$, where $\beta \in \mathbb{C}$, and two $T(q)$-Galois objects $A_{(1, \beta)}$ and $A_{(1, \mu)}$ are isomorphic if and only if $\beta = \mu$.

By Theorem 5.13, if there is a semisimple module category of rank 1 over $\mathcal{C} = T(q)^{\mathcal{M}} \rtimes \mathbb{Z}/(N)$, then it must be a $T(q)^{\mathcal{M}}$-module category $\mathbb{Z}/(N)$-invariant.

Suppose that $A_{(1, \beta)}$ is $\mathbb{Z}/(N)$-invariant. Since $A_{(r, \mu)} \mathbb{Z}/(N) A_{(1, \beta)} \cong A_{(r, \mu + \beta)}$, we have that $\mu = 0$. If the action is associated to a pair $(r, \mu)$, where $\mu \neq 0$, then the category $\mathcal{C}$ does not admit any fibre functor, that is, it is not the category of comodules of a Hopf algebra.

However, since every simple object is invertible, the Perron–Frobenius dimension of the simple objects is 1. So, by [13, Proposition 2.7], the tensor category $T(q)^{\mathcal{M}} \rtimes \mathbb{Z}/(N)$ is equivalent to the category of representations of a quasi-Hopf algebra.

Note that the tensor category $(T(q)^{\mathcal{M}})^G$ has at least one fibre functor for every group and every group action. In fact, since the forgetful functor $U : T(q)^{\mathcal{M}}^G \to T(q)^{\mathcal{M}}$ is monoidal, then the composition with the fibre functor of $T(q)^{\mathcal{M}}$ gives a fibre functor on $(T(q)^{\mathcal{M}})^G$.

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References


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