Decidability of Order-Based Modal Logics

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Abstract

Decidability of the validity problem is established for a family of many-valued modal logics, notably modal logics based on Gödel logics, where propositional connectives are evaluated locally at worlds according to the order of values in a complete sublattice of the real unit interval, and box and diamond modalities are evaluated as infima and suprema of values in (many-valued) Kripke frames. When the sublattice is infinite and the language is sufficiently expressive, the standard semantics for such a logic lacks the finite model property. It is shown here, however, that the finite model property holds for a new equivalent semantics for the same logic. Decidability and PSPACE-completeness of the validity problem follows from this property, given certain regularity conditions on the order of the sublattice. Decidability and co-NP-completeness of the validity problem are also

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established for $S5$ versions of the logics that coincide with one-variable fragments of first-order many-valued logics. In particular, a first proof is provided of the decidability and co-NP-completeness for the validity problem of the one-variable fragment of first-order Gödel logic.

Keywords: Modal logics, Many-valued logics, Gödel logics, One-Variable Fragments, Decidability, Complexity, Finite Model Property

1. Introduction

Many-valued modal logics extend the Kripke frame setting of classical modal logic with a many-valued semantics at each world and a many-valued or crisp (Boolean-valued) accessibility relation to model modal notions such as necessity, belief, and spatio-temporal relations in the presence of uncertainty, possibility, or vagueness. Applications include modelling fuzzy belief [16, 21], spatial reasoning with vague predicates [32], many-valued tense logics [12], fuzzy similarity measures [17], and fuzzy description logics, interpreted, analogously to the classical case, as many-valued multi-modal logics (see, e.g., [5, 20, 34]).

Quite general approaches to many-valued modal logics, focussing largely on decidability and axiomatization for finite-valued modal logics, are described in [6, 14, 15, 29]. For modal logics based on an infinite-valued semantics, typically over the real unit interval $[0, 1]$, two core families can be identified. Many-valued modal logics of “magnitude” are based on a semantics related to Łukasiewicz infinite-valued logic with propositional connectives interpreted by continuous functions over real numbers [6, 18, 22]. Typical many-valued modal logics of the second family are based instead on the semantics of infinite-valued Gödel logics [9, 10, 13, 18, 26]. The standard infinite-valued Gödel logic (also known as Gödel-Dummett logic) interprets truth values as elements of $[0, 1]$, conjunction and disjunction connectives as minimum and maximum, respectively, and implication $x \rightarrow y$ as $y$ for $x > y$ and 1 otherwise. Modal operators $\Box$ and $\Diamond$ (not inter-definable in this setting) can then be interpreted as infima and suprema of values calculated at accessible worlds. More generally, “order-based” modal logics may be based on a complete sublattice of $\langle [0, 1], \wedge, \vee, 0, 1 \rangle$ with additional operations depending only on the given order.

Propositional Gödel logic has been studied intensively both as a fundamental “t-norm based” fuzzy logic [18, 27] and as an intermediate (or superintuitionistic) logic, obtained as an extension of any axiomatization of propositional intuitionistic logic with the prelinearity axiom schema $(\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)$. The many-
valued modal logics considered in this paper diverge considerably, however, from
the modal intermediate logics investigated in [35] (and elsewhere), which use two
accessibility relations for Kripke models, one for the modal operators and one for
the intuitionistic connectives. We remark also that, unlike the operations added
to infinite-valued logics in [11, 19], which represent truth stressers such as “very
true” or “classically true”, the modalities considered here cannot be interpreted
simply as unary connectives on the real unit interval \([0, 1]\).

The first main contribution of this paper is to establish PSPACE-completeness
(matching the complexity of the classical modal logic \(\mathbf{K}\) [24]) of the validity prob-
lem for Gödel modal logics over \([0, 1]\), possibly with additional order-invariant
connectives, and for other order-based modal logics over complete sublattices of
\([0, 1]\) satisfying certain local regularity conditions. Such sublattices include cases
order-isomorphic to the positive integers with an added upper bound \(\mathbb{N} \cup \{\infty\}\)
and the negative integers with an added lower bound \(\{-n : n \in \mathbb{N}\} \cup \{-\infty\}\).

The finite model property typically fails even for the box and diamond fragments
of these logics. Decidability and PSPACE-completeness of the validity problem
for these fragments in the case of Gödel modal logics over \([0, 1]\) was established
in [26] using analytic Gentzen-style proof systems, but this methodology does not
seem to extend easily to the full logics. Here, an alternative Kripke semantics is
provided for an order-based modal logic that not only has the same valid formulas
as the original semantics, but also admits the finite model property. The key idea
of this new semantics is to restrict evaluations of modal formulas at a world to a
particular set of truth values.

The second main contribution of the paper is to establish decidability and co-
NP-completeness results for the validity problem of crisp order-based “S5” logics:
order-based modal logics where accessibility is an equivalence relation. Such logi-
cs may be interpreted also as one-variable fragments of first-order many-valued
logics. In particular, the open decidability problem for validity in the one-variable
fragment of first-order Gödel logic (see, e.g., [18, Chapter 9, Problem 13]) is
answered positively and shown to be co-NP-complete. This result matches the
complexity of the one-variable fragments of classical first-order logic (equiva-
ently, S5) and first-order Łukasiewicz logic (see [18]), and contrasts with the co-
NEXPTIME-completeness of the one-variable fragment of first-order intuitionis-
tic logic (equivalently, the intuitionistic modal logic \(\text{MIPC}\) [25].

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2. Order-Based Modal Logics

We consider “order-based” modal logics where propositional connectives are interpreted at individual worlds in an algebra consisting of a complete sublattice of $\langle [0, 1], \land, \lor, 0, 1 \rangle$ with operations defined based only on the order. Modalities $\Box$ and $\Diamond$ are defined using infima and suprema, respectively, according to either a (crisp or Boolean-valued) binary relation on the set of worlds or a binary mapping (many-valued relation) from worlds to values of the algebra. For convenience, we consider only finite algebraic languages, noting that to decide the validity of a formula we may in any case restrict to the language containing only operation symbols occurring in that formula.

We reserve the symbols $\Rightarrow$, $\&$, $\sim$, and $\approx$ to denote implication, conjunction, negation, and equality, respectively, in classical first-order logic. We also recall an appropriate notion of first-order definability of operations for algebraic structures.

Let $L$ be a finite algebraic language, $A$ an algebra for $L$, and $L'$ a sublanguage of $L$. An operation $f : A^n \to A$ is defined in $A$ by a first-order $L'$-formula $F(x_1, \ldots, x_n, y)$ with free variables $x_1, \ldots, x_n, y$ if for all $a_1, \ldots, a_n, b \in A$, $A \models F(a_1, \ldots, a_n, b) \iff f(a_1, \ldots, a_n) = b$.

2.1. Order-Based Algebras

Let $L$ be a finite algebraic language that includes the binary operation symbols $\land$ and $\lor$ and constant symbols $\bar{0}$ and $\bar{1}$ (to be interpreted by the usual lattice operations), and denote the finite set of constants (nullary operation symbols) of this language by $C_L$. An algebra $A$ for $L$ will be called order-based if it satisfies the following conditions:

(1) $\langle A, \land^A, \lor^A, 0, 1 \rangle$ is a complete sublattice of $\langle [0, 1], \min, \max, 0, 1 \rangle$; i.e., $\{0, 1\} \subseteq A \subseteq [0, 1]$ and for all $B \subseteq A$, $\land^{[0,1]} B$ and $\lor^{[0,1]} B$ belong to $A$.

(2) For each operation symbol $\star$ of $L$, the operation $\star^A$ is definable in $A$ by a quantifier-free first-order formula in the algebraic language with only $\land, \lor$, and constants from $C_L$.

We also let $C_L^A$ denote the finite set of constant operations $\{c^A : c \in C_L\}$, and define $R(A)$ and $L(A)$ to be the sets of right and left accumulation points, respectively, of $A$ in the usual topology inherited from $[0, 1]$; that is,
\[ a \in R(A) \text{ iff } \text{there is a } c \in A \text{ such that } a \triangleleft_{A} c \text{ and for all such } c, \text{there is an } e \in A \text{ such that } a \triangleleft_{A} e \triangleleft_{A} c. \]

\[ b \in L(A) \text{ iff } \text{there is a } d \in A \text{ such that } d \triangleleft_{A} b, \text{ and for all such } d, \text{there is an } f \in A \text{ such that } d \triangleleft_{A} f \triangleleft_{A} b. \]

Note that, because \( A \) is a chain, an implication operation \( \rightarrow_{A} \) may always be introduced as the residual of \( \wedge_{A} \):

\[
a \rightarrow_{A} b = \bigvee_{A} \{ c \in A : c \wedge_{A} a \leq_{A} b \} = \begin{cases} 1 & \text{if } a \leq_{A} b \\ b & \text{otherwise.} \end{cases}
\]

Let \( s \leq t \) stand for \( s \wedge t \approx s \) and let \( s < t \) stand for \( (s \leq t) \& \neg (s \approx t) \). Then the implication operation \( \rightarrow_{A} \) is definable in \( A \) by the quantifier-free first-order formula

\[
F^\rightarrow(x, y, z) = ( (x \leq y) \Rightarrow (z \approx \bar{1}) ) \& ( (y < x) \Rightarrow (z \approx y) ).
\]

That is, for all \( a, b, c \in A, \)

\[ A \models F^\rightarrow(a, b, c) \iff a \rightarrow_{A} b = c. \]

In this paper, the connective \( \rightarrow_{A} \) will always be interpreted by \( \rightarrow_{A} \) in \( A \). We will also make use of the negation connective \( \neg \varphi := \varphi \rightarrow \bar{0} \), which is interpreted by the unary operation

\[
\neg_{A} a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise.} \end{cases}
\]

Examples of other useful operations (see, e.g., [1]) covered by the order-based approach are the globalization and Nabla operators

\[
\Delta_{A} a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \nabla_{A} a = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{otherwise}, \end{cases}
\]

definable in \( A \) (noting also that \( \nabla_{A} a = \neg_{A} \neg_{A} a \)), by

\[
F^\Delta(x, y) = ( (x \approx \bar{1}) \Rightarrow (y \approx \bar{1}) ) \& ( (x < \bar{1}) \Rightarrow (y \approx \bar{0}) )
\]
\[
F^\nabla(x, y) = ( (x \approx \bar{0}) \Rightarrow (y \approx \bar{0}) ) \& ( (\bar{0} < x) \Rightarrow (y \approx \bar{1}) )
\]
and the dual-implication connective (the residual of \( \vee_{A} \))

\[
a \leftarrow_{A} b = \bigwedge_{A} \{ c \in A : b \leq_{A} a \vee_{A} c \} = \begin{cases} 0 & \text{if } b \leq_{A} a \\ b & \text{otherwise,} \end{cases}
\]
definable in $A$ by

$$F(x, y, z) = ((y \leq x) \Rightarrow (z \approx \bar{0})) \& ((x < y) \Rightarrow (z \approx y)).$$

For the remainder of this work, we will omit the superscript $A$ when the algebra or order is clear from the context.

### 2.2. Many-Valued Kripke Semantics

Let us fix a language $L$ including the operational symbols $\bar{1}, \bar{0}, \land, \lor,$ and $\to$, and an order-based algebra $A$ for $L$. We define order-based modal logics $K(A)^C$ and $K(A)$ based on standard (crisp) Kripke frames and Kripke frames with an accessibility relation taking values in $A$, respectively.

An $A$-frame is a pair $\mathcal{F} = \langle W, R \rangle$ such that $W$ is a non-empty set of worlds and $R: W \times W \to A$ is an $A$-accessibility relation on $W$. If $R_{xy} \in \{0, 1\}$ for all $x, y \in W$, then $R$ is crisp and $\mathcal{F}$ is called a crisp $A$-frame or simply a frame. In this case, we often write $R \subseteq W \times W$ and $R_{xy}$ to mean $R_{xy} = 1$.

Now let $Fm$ be the set of formulas, denoted by $\varphi, \psi, \chi, \ldots$, of the language $L$ with additional unary operation symbols (modal connectives) $\Box$ and $\Diamond$, defined inductively over a countably infinite set $\text{Var}$ of propositional variables, denoted by $p, q, \ldots$. We call formulas of the form $\Box \varphi$ and $\Diamond \varphi$ box-formulas and diamond-formulas, respectively. Subformulas are defined as usual, and the length of a formula $\varphi$, denoted by $\ell(\varphi)$, is the total number of occurrences of subformulas in $\varphi$. We also let $\text{Var}(\varphi)$ denote the set of variables occurring in the formula $\varphi$.

A $K(A)$-model is a triple $\mathcal{M} = \langle W, R, V \rangle$ such that $\langle W, R \rangle$ is an $A$-frame and $V: \text{Var} \times W \to A$ is a mapping, called a valuation, that is extended to $V: Fm \times W \to A$ by

$$V(\ast(\varphi_1, \ldots, \varphi_n), x) = \ast(V(\varphi_1, x), \ldots, V(\varphi_n, x))$$

for each $n$-ary operation symbol $\ast$ of $L$, and

$$V(\Box \varphi, x) = \bigwedge \{R_{xy} \to V(\varphi, y) : y \in W\}$$
$$V(\Diamond \varphi, x) = \bigvee \{R_{xy} \land V(\varphi, y) : y \in W\}.$$ 

A $K(A)^C$-model satisfies the extra condition that $\langle W, R \rangle$ is a crisp $A$-frame. In this case, the conditions for $\Box$ and $\Diamond$ simplify to

$$V(\Box \varphi, x) = \bigwedge \{V(\varphi, y) : R_{xy}\}$$
$$V(\Diamond \varphi, x) = \bigvee \{V(\varphi, y) : R_{xy}\}.$$
A formula \( \varphi \in \text{Fm} \) will be called \textit{valid} in a \( \mathbf{K}(A) \)-model \( \mathfrak{M} = \langle W, R, V \rangle \) if \((V(\varphi, x) = 1 \) for all \( x \in W \). If \( \varphi \) is valid in all \( \mathbf{L} \)-models for some logic \( \mathbf{L} \), then \( \varphi \) is said to be \( \mathbf{L} \)-valid, written \( \models_{\mathbf{L}} \varphi \).

We now introduce some useful notation and terminology. A subset \( \Sigma \subseteq \text{Fm} \) will be called a \textit{fragment} if it contains all constants in \( \mathcal{C} \) and is closed with respect to taking subformulas. For a formula \( \varphi \in \text{Fm} \), we let \( \Sigma(\varphi) \) be the smallest (always finite) fragment containing \( \varphi \). Also, for any \( \mathbf{K}(A) \)-model \( \mathfrak{M} = \langle W, R, V \rangle \), set \( X \subseteq W \), and fragment \( \Sigma \subseteq \text{Fm} \), we let

\[
V[\Sigma, X] = \{ V(\varphi, x) : \varphi \in \Sigma \text{ and } x \in X \}.
\]

We shorten \( V[\Sigma, \{x\}] \) to \( V[\Sigma, x] \). For \( \Sigma \subseteq \text{Fm} \), we let \( \Sigma_{\square} \) and \( \Sigma_{\lozenge} \) be the sets of all box-formulas in \( \Sigma \) and diamond-formulas in \( \Sigma \), respectively.

Given a linearly ordered set \( \langle P, \leq \rangle \) and \( C \subseteq P \), a map \( h: P \rightarrow P \) will be called a \textit{C-order embedding} if it is an order-preserving embedding (i.e., \( a \leq b \) if and only if \( h(a) \leq h(b) \) for all \( a, b \in P \)) satisfying \( h(c) = c \) for all \( c \in C \). \( h \) will be called \textit{B-complete} for \( B \subseteq P \) if whenever \( \bigvee D \in B \) or \( \bigwedge D \in B \) for some \( D \subseteq P \), respectively,

\[
h(\bigvee D) = \bigvee h[D] \quad \text{or} \quad h(\bigwedge D) = \bigwedge h[D].
\]

The following lemma establishes the critical property of order-based modal logics for our purposes. Namely, it is only the order of the values taken by variables and the accessibility relation between worlds that plays a role in determining the values of formulas and checking validity.

\textbf{Lemma 1.} Let \( \mathfrak{M} = \langle W, R, V \rangle \) be a \( \mathbf{K}(A) \)-model and \( \Sigma \subseteq \text{Fm} \) a fragment, and let \( h: A \rightarrow A \) be a \( V[\Sigma_{\square} \cup \Sigma_{\lozenge}, W] \)-complete \( \mathcal{C} \)-order embedding. Consider the \( \mathbf{K}(A) \)-model \( \widehat{\mathfrak{M}} = \langle W, \widehat{R}, \widehat{V} \rangle \) with \( \widehat{R}xy = h(Rxy) \) and \( \widehat{V}(p, x) = h(V(p, x)) \) for all \( p \in \text{Var} \) and \( x, y \in W \). Then for all \( \varphi \in \Sigma \) and \( x \in W \):

\[
\widehat{V}(\varphi, x) = h(V(\varphi, x)).
\]

\textbf{Proof.} We proceed by induction on \( \ell(\varphi) \). The case \( \varphi \in \text{Var} \cup \mathcal{C} \) follows from the definition of \( \widehat{V} \). For the induction step, suppose that \( \varphi = *(\varphi_1, \ldots, \varphi_n) \) for some operation symbol \( * \) of \( \mathcal{L} \) and \( \varphi_1, \ldots, \varphi_n \in \Sigma \). Recall that \( * \) is definable in \( A \) by some quantifier-free first-order formula \( F^*(x_1, \ldots, x_n, y) \) in the first-order language with \( \land, \lor, \) and constants from \( \mathcal{C} \), i.e.

\[
*(a_1, \ldots, a_n) = b \quad \iff \quad A \models F^*(a_1, \ldots, a_n, b).
\]
Because $F^*(x_1,\ldots,x_n,y)$ is quantifier-free,

$$A \models F^*(a_1,\ldots,a_n,b) \iff A \models F^*(h(a_1),\ldots,h(a_n),h(b)).$$

So we may also conclude

$$\star(h(a_1),\ldots,h(a_n)) = h(\star(a_1,\ldots,a_n)).$$

Hence for all $x \in W$, using the induction hypothesis for the step from (1) to (2):

$$\hat{V}(\star(\varphi_1,\ldots,\varphi_n),x) = \star(\hat{V}(\varphi_1,x),\ldots,\hat{V}(\varphi_n,x)) \quad (1)$$
$$= \star(h(V(\varphi_1),x),\ldots,h(V(\varphi_n),x)) \quad (2)$$
$$= h(\star(V(\varphi_1),\ldots,V(\varphi_n),x)) \quad (3)$$
$$= h(V(\star(\varphi_1,\ldots,\varphi_n),x)). \quad (4)$$

If $\varphi = \lozenge\psi$ for some $\psi \in \Sigma$, then we obtain for all $x \in W$:

$$\hat{V}(\lozenge\psi,x) = \bigvee \{\hat{R}xy \land \hat{V}(\psi,y) : y \in W\} \quad (5)$$
$$= \bigvee \{h(Rxy) \land h(V(\psi,y)) : y \in W\} \quad (6)$$
$$= \bigvee \{h(Rxy) \land V(\psi,y) : y \in W\} \quad (7)$$
$$= h(\bigvee \{Rxy \land V(\psi,y) : y \in W\}) \quad (8)$$
$$= h(V(\lozenge\psi,x)). \quad (9)$$

(5) to (6) follows from the definition of $\hat{R}$ and the induction hypothesis, (6) to (7) follows because $h$ is an order embedding, and (7) to (8) follows because $h$ is $V[\Sigma\Box \cup \Sigma\lozenge,W]$-complete and $\bigvee \{Rxy \land V(\psi,y) : y \in W\} = V(\lozenge\psi,x) \in V[\Sigma\lozenge,W]$. The case $\varphi = \Box\psi$ is very similar.

We now consider many-valued analogues of some useful notions and results from classical modal logic (see, e.g., [4]). For an $A$-frame $\langle W,R \rangle$, we define the crisp relation $R^+$ as follows:

$$R^+ = \{(x,y) \in W \times W : Rxy > 0\}, \quad R^+[x] = \{y \in W : R^+xy\} \quad \text{for} \ x \in W.$$ 

Let $\mathfrak{M} = \langle W,R,V \rangle$ be a $K(A)$-model. We call $\mathfrak{M}' = \langle W',R',V' \rangle$ a $K(A)$-submodel of $\mathfrak{M}$, written $\mathfrak{M}' \subseteq \mathfrak{M}$, if $W' \subseteq W$ and $R'$ and $V'$ are the restrictions to $W'$ of $R$ and $V$, respectively. In particular, given $x \in W$, the $K(A)$-submodel of $\mathfrak{M}$ generated by $x$ is the smallest $K(A)$-submodel $\mathfrak{M}' = \langle W',R',V' \rangle$ of $\mathfrak{M}$ satisfying $x \in W'$ and for all $y \in W'$, $z \in R^+[y]$ implies $z \in W'$.
Lemma 2. Let \( \mathcal{M} = (W, R, V) \) be a \( \mathsf{K(\mathsf{A})} \)-model and \( \hat{\mathcal{M}} = (\hat{W}, \hat{R}, \hat{V}) \) a generated \( \mathsf{K(\mathsf{A})} \)-submodel of \( \mathcal{M} \). Then \( \hat{V}(\varphi, x) = V(\varphi, x) \) for all \( x \in \hat{W} \) and \( \varphi \in \text{Fm} \).

Proof. We proceed by induction on \( \ell(\varphi) \). The base case is trivial for any submodel of \( \mathcal{M} \), so also for \( \hat{\mathcal{M}} \). For the induction step, the case where \( \varphi = \ast(\varphi_1, \ldots, \varphi_n) \) follows immediately using the induction hypothesis.

Suppose now that \( \varphi = \Box \psi \). Fix \( x \in \hat{W} \) and note that for any \( y \in W \setminus \hat{W} \), we have \( Rxy = 0 \). Observe also that \( 0 \rightarrow a = 1 \) for all \( a \in A \). Hence, excluding all worlds \( y \in W \) such that \( Rxy = 0 \) does not change the value of \( \bigwedge \{ Rxy \rightarrow V(\psi, y) : y \in W \} \). So, using the induction hypothesis,

\[
V(\Box \psi, x) = \bigwedge \{ Rxy \rightarrow V(\psi, y) : y \in \hat{W} \}
= \bigwedge \{ \hat{R}xy \rightarrow \hat{V}(\psi, y) : y \in \hat{W} \}
= \hat{V}(\Box \psi, x).
\]

The case where \( \varphi = \Diamond \psi \) is very similar. \( \Box \)

Following the usual terminology of modal logic, a tree is defined as a relational structure \( (T, S) \) such that (i) \( S \subseteq T^2 \) is irreflexive, (ii) there exists a unique root \( x_0 \in T \) satisfying \( S^*x_0x \) for all \( x \in T \) where \( S^* \) is the reflexive transitive closure of \( S \), (iii) for each \( x \in T \setminus \{ x_0 \} \), there is a unique \( x' \in T \) such that \( Sx'x \). A tree \( (T, S) \) has height \( m \in \mathbb{N} \) if \( m = \max \{|\{ y \in T : S^*y x \}| : x \in T \} \). A \( \mathsf{K(\mathsf{A})} \)-model \( \mathcal{M} = (W, R, V) \) is called a \( \mathsf{K(\mathsf{A})} \)-tree-model if \( (W, R^+) \) is a tree, and has finite height \( \text{hgl}(\mathcal{M}) = m \) if \( (W, R^+) \) has height \( m \).

Lemma 3. Let \( \mathcal{M} = (W, R, V) \) be a \( \mathsf{K(\mathsf{A})} \)-model, \( x_0 \in W \), and \( k \in \mathbb{N} \). Then there exists a \( \mathsf{K(\mathsf{A})} \)-tree-model \( \hat{\mathcal{M}} = (\hat{W}, \hat{R}, \hat{V}) \) with root \( \hat{x}_0 \) and \( \text{hgl}(\hat{\mathcal{M}}) \leq k \) such that \( \hat{V}(\varphi, \hat{x}_0) = V(\varphi, x_0) \) for all \( \varphi \in \text{Fm} \) with \( \ell(\varphi) \leq k \). Moreover, if \( \mathcal{M} \) is a \( \mathsf{K(\mathsf{A})}^c \)-model, then so is \( \hat{\mathcal{M}} \).

Proof. Consider the \( \mathsf{K(\mathsf{A})} \)-model \( \mathcal{M}' = (W', R', V') \) obtained by “unravelling” at the world \( x_0 \); i.e., for all \( n \in \mathbb{N} \) (noting that \( 0 \in \mathbb{N} \)),

\[
W' = \bigcup_{n \in \mathbb{N}} \{(x_0, \ldots, x_n) \in W^{n+1} : R^+x_ix_{i+1} \text{ for } i < n \}
\]

\[
R'yz = \begin{cases} R_{x_nx_{n+1}} & \text{if } y = (x_0, \ldots, x_n), \ z = (x_0, \ldots, x_{n+1}) \\ 0 & \text{otherwise} \end{cases}
\]

\[
V'(p, (x_0, \ldots, x_n)) = V(p, x_n).
\]
Clearly, \( \mathcal{M} \) is a K(A)-tree-model with root \( \tilde{x}_0 = (x_0) \). Now let \( \hat{\mathcal{M}} = \langle \hat{W}, \hat{R}, \hat{V} \rangle \) be the K(A)-tree-submodel of \( \mathcal{M} \) defined by cutting \( \mathcal{M} \) at depth \( k \); i.e., let \( \hat{W} = \{(x_0, \ldots, x_n) \in W' : n \leq k \} \) and let \( \hat{R} \) and \( \hat{V} \) be the restrictions of \( R' \) and \( V' \) to \( \hat{W} \times \hat{W} \) and \( Var \times \hat{W} \), respectively. A straightforward induction on \( \ell(\varphi) \) shows that for all \( \varphi \in \mathcal{Fm} \) and \( n \in \mathbb{N} \) such that \( \ell(\varphi) \leq k - n \), \( \hat{V}(\varphi, (x_0, \ldots, x_n)) = V(\varphi, x_n) \). In particular, \( \hat{V}(\varphi, \tilde{x}_0) = V(\varphi, x_0) \) for all \( \varphi \in \mathcal{Fm} \) with \( \ell(\varphi) \leq k \).

2.3. Gödel Modal Logics

The “Gödel modal logics” \( \mathcal{GK} \) and \( \mathcal{GK}^C \) studied in [6, 9, 10, 26] are \( \mathcal{K}(G) \) and \( \mathcal{K}(G)^C \), respectively, based on the standard infinite-valued Gödel algebra:

\[
\mathcal{G} = \langle [0, 1], \land, \lor, \rightarrow, 0, 1 \rangle.
\]

Axiomatizations of the box and diamond fragments of \( \mathcal{GK} \) are obtained in [9] as extensions of an axiomatization of Gödel logic (intuitionistic logic plus the prelinearity axiom schema \((\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)\)) with, respectively,

\[
\begin{align*}
\neg\neg\square \varphi & \rightarrow \square \neg\neg \varphi \\
\square (\varphi \rightarrow \psi) & \rightarrow (\square \varphi \rightarrow \square \psi) \quad \text{and} \quad \diamond(\varphi \lor \psi) & \rightarrow (\diamond \varphi \lor \diamond \psi) \\
\varphi & \rightarrow (\diamond \varphi \rightarrow \square \psi) \\
\neg\diamond0 & \rightarrow \diamond \neg\neg \psi
\end{align*}
\]

An axiomatization of the full logic \( \mathcal{GK} \) is obtained in [10] by extending the union of these axiomatizations with the Fischer Servi axioms (see [33])

\[
\begin{align*}
\diamond(\varphi \rightarrow \psi) & \rightarrow (\square \varphi \rightarrow \diamond \psi) \\
(\diamond \varphi \rightarrow \square \psi) & \rightarrow \square (\varphi \rightarrow \psi).
\end{align*}
\]

It is also shown in [10] that \( \mathcal{GK} \) coincides with the extension of the intuitionistic modal logic \( \mathcal{IK} \) with the prelinearity axiom schema \((\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)\).

No axiomatization has yet been found for the full logic \( \mathcal{GK}^C \). However, the box fragment of \( \mathcal{GK}^C \) coincides with the box fragment of \( \mathcal{GK} \) [9], and the diamond fragment of \( \mathcal{GK}^C \) is axiomatized in [26] as an extension of the diamond fragment of \( \mathcal{GK} \) with

\[
\varphi \lor (\psi_1 \rightarrow \psi_2) \rightarrow (\diamond \varphi \lor (\diamond \psi_1 \rightarrow \diamond \psi_2)).
\]
More generally, we may consider the family of Gödel modal logics $K(A)$ and $K(A)^C$ where $A$ is a complete subalgebra of $G$: in particular, when $A$ is $G_\downarrow = (G_\downarrow, \land, \lor, \to, 0, 1)$ or $G_\uparrow = (G_\downarrow, \land, \lor, \to, 0, 1)$ with
\[ G_\downarrow = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \quad \text{and} \quad G_\uparrow = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{1\}. \]
Clearly, order-based algebras with universes $G_\downarrow$ and $G_\uparrow$ are isomorphic to algebras with universes $\{-n : n \in \mathbb{N}\} \cup \{-\infty\}$ and $\mathbb{N} \cup \{\infty\}$, respectively.

It is not hard to show (see below) that for finite $A$, the sets of valid formulas of $K(A)$ and $K(A)^C$ depend only on the cardinality of $A$ and are decidable. Note, moreover, that although all infinite subalgebras of $G$ produce the same set of valid propositional formulas [13], there are countably infinitely many different infinite-valued first-order Gödel logics (considered as sets of valid formulas) [2]. This result holds also for Gödel modal logics.

**Proposition 4.** There are countably infinitely many different infinite-valued Gödel modal logics (considered as sets of valid formulas).

**Proof.** Note first that, by the result of [2], there can be at most countably many such logics, as each Gödel modal logic can be interpreted as a fragment of a corresponding first-order Gödel logic. To show that there are infinitely many different Gödel modal logics, let us fix, for each $n \in \mathbb{N}^+$, a complete subalgebra $A_n$ of $G$ with exactly $n$ right accumulation points. We define the formula
\[ \varphi(p, q) = (\Box(p \to q) \land \Box((p \to q) \to q)) \to ((\Box p \to q) \to q), \]
and then for $1 \leq n \in \mathbb{N}$, the formulas
\[ \varphi_n(p_1, q_1, \ldots, p_n, q_n) = \bigwedge_{i=1}^{n-1} ((q_{i+1} \to q_i) \to q_i) \to \bigvee_{i=1}^n \varphi(p_i, q_i). \]
It follows (we leave the reader to check the details) that $\varphi_{n+1}$ is $K(A_n)$-valid and $K(A_n)^C$-valid, but neither $K(A_m)$-valid nor $K(A_m)^C$-valid for any $m > n + 1$. \[\square\]

The logics $K(G)$, $K(G_\downarrow)$, $K(G_\uparrow)$ and their crisp counterparts are all distinct. The formula $\Box \neg \neg p \to \neg \neg \Box p$ is valid in the logics based on $G_\uparrow$, but not in those based on $G$ or $G_\downarrow$. To see this, note that 0 is an accumulation point in $[0, 1]$ and $G_\downarrow$ (but not in $G_\uparrow$); hence for these sets there is an infinite descending sequence of values $(a_i)_{i \in I}$ with limit 0, giving $\neg \neg a_i = 1$ for each $i \in I$ and $\inf_{i \in I} \neg \neg a_i = 1$, while $\inf_{i \in I} a_i = 0 \neq 0$ (see the proof of Theorem 7). Similarly, $(\Diamond p \to \Diamond q) \to (\neg \Diamond q \lor \Diamond(p \to q))$ is valid in the logics based on $G_\uparrow$ but not those based on $G$. Moreover, the formula $\neg \neg \Diamond p \to \Diamond \neg \neg p$ is valid in any of the crisp logics, but not in the non-crisp versions.
2.4. The Finite Model Property

Let us call an $L$-model for a logic $L$ countable or finite if its set of worlds is countable or finite, respectively. We say that a logic $L$ has the finite model property if validity in the logic coincides with validity in all finite $L$-models. Observe first that if the underlying algebra of an order-based modal logic is finite, then the logic has the finite model property.

**Lemma 5.** If $A$ is a finite order-based algebra, then $K(A)$ and $K(A)^C$ have the finite model property.

**Proof.** By Lemma 3, it suffices to show that for any finite fragment $\Sigma \subseteq Fm$ and $K(A)$-tree-model $M = \langle W, R, V \rangle$ of finite height with root $x$, there is a finite $K(A)$-tree-model $\widehat{M} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle \subseteq M$ with root $x$ such that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \Sigma$. We prove this claim by induction on $\text{hg}(\mathcal{M})$. For the base case, $W = \{x\}$ and we let $\widehat{M} = M$.

For the induction step, consider for each $y \in R^+[x]$, the submodel $M_y = \langle W_y, R_y, V_y \rangle$ of $M$ generated by $y$. Each $M_y$ is a $K(A)$-tree-model of finite height with root $y$ and $\text{hg}(M_y) < \text{hg}(M)$. Hence, by the induction hypothesis, for each $y \in R^+[x]$, there is a finite $K(A)$-tree-model $\widehat{M}_y = \langle \widehat{W}_y, \widehat{R}_y, \widehat{V}_y \rangle \subseteq M_y \subseteq M$ with root $y \in \widehat{W}_y$ such that for all $\varphi \in \Sigma$, by Lemma 2, $\widehat{V}_y(\varphi, y) = V_y(\varphi, y) = V(\varphi, y)$.

Because $A$ is finite, we can now choose for each $\varphi \in \Sigma_\Box \cup \Sigma_\Diamond$ a world $y_\varphi$ such that $V(\varphi, x) = \widehat{V}_y(\psi, y_\varphi)$ where $\varphi = \Box \psi$ or $\varphi = \Diamond \psi$. Define the finite set $Y = \{y_\varphi \in R^+[x] : \varphi \in \Sigma_\Box \cup \Sigma_\Diamond\}$. We let $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$ where

$$\widehat{W} = \{x\} \cup \bigcup_{y \in Y} \widehat{W}_y,$$

and $\widehat{R}$ and $\widehat{V}$ are $R$ and $V$, respectively, restricted to $\widehat{W}$. An easy induction on $\ell(\varphi)$ establishes that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \Sigma$. $\square$

We are also able to establish the finite model property for certain cases where the underlying (infinite) algebra is $G_\uparrow$.

**Theorem 6.** $K(G_\uparrow)$ and $K(G_\uparrow)^C$ have the finite model property.

**Proof.** By Lemmas 3 and 5, it suffices to show that if $\varphi \in Fm$ is not valid in some $K(G_\uparrow)$-tree-model $M$ of finite height, then there is a finite subalgebra $B$ of $G_\uparrow$ and a $K(B)$-model $\widehat{M}$ (that is crisp if $M$ is crisp) such that $\varphi$ is not valid in $\widehat{M}$.
Suppose that $\beta = V(\varphi, x) < 1$ for some $K(G_\uparrow)$-tree-model of finite height $\mathfrak{M} = \langle W, R, V \rangle$ with root $x$. Let $B$ be the finite subalgebra of $G_\uparrow$ with universe $(G_\uparrow \cap [0, \beta]) \cup \{1\}$ and consider $h : G_\uparrow \to B$ defined by

$$h(a) = \begin{cases} a & \text{if } a \leq \beta \\ 1 & \text{otherwise.} \end{cases}$$

We define a $K(B)$-model $\hat{\mathfrak{M}} = \langle W, \hat{R}, \hat{V} \rangle$ (that is crisp if $\mathfrak{M}$ is crisp) as follows. Let $\hat{R}_{yz} = h(R_{yz})$ for all $y, z \in W$ and $\hat{V}(p, y) = h(V(p, y))$ for all $y \in W$ and $p \in \text{Var}$. We prove that $\hat{V}(\psi, y) = h(V(\psi, y))$ for all $y \in W$ and $\psi \in \text{Fm}$ by induction on $\ell(\psi)$. The base case follows by definition (recalling that the only constants are $\bar{0}$ and $\bar{1}$). For the induction step, the propositional cases follow by observing that $h$ is a Heyting algebra homomorphism (i.e., preserves the operations $\land, \lor, \to, \bar{0}$, and $\bar{1}$). If $\psi = \Diamond \chi$, then

$$\hat{V}(\Diamond \chi, y) = \bigvee \{\hat{R}_{yz} \land \hat{V}(\chi, z) : z \in W\}$$

$$= \bigvee \{h(R_{yz}) \land h(V(\chi, z)) : z \in W\}$$

$$= \bigvee \{h(R_{yz} \land V(\chi, z)) : z \in W\}$$

$$= h(\bigvee \{R_{yz} \land V(\chi, z) : z \in W\})$$

$$= h(V(\Diamond \chi, y)).$$

The step from (10) to (11) follows using the induction hypothesis and the step from (11) to (12) follows because $h$ is a Heyting algebra homomorphism. For the step from (12) to (13), note that for $\bigvee \{R_{yz} \land V(\chi, z) : z \in W\} \leq \beta$, the equality is immediate. Otherwise, $R_{yz} \land V(\chi, z) > \beta$ for some $z \in W$ and $h(R_{yz} \land V(\chi, z)) = 1$, so $h(\bigvee \{R_{yz} \land V(\chi, z) : z \in W\}) = 1 = \bigvee \{h(R_{yz} \land V(\chi, z)) : z \in W\}$. The case of $\psi = \Box \chi$ is very similar.

Hence $\hat{V}(\varphi, x) = h(V(\varphi, x)) = h(\beta) = \beta < 1$ as required. $\square$

The finite model property does not hold, however, for Gödel modal logics with universe $[0, 1]$ or $G_\downarrow$, or even $G_\uparrow$ if we add also the connective $\Delta$ to the language. The problem in these cases stems from the existence of accumulation points in the universe of truth values. If infinitely many worlds are accessible from a world $x$, then the value taken by a formula $\Box \varphi$ (or $\Diamond \varphi$) at $x$ will be the infimum (supremum) of values calculated from values of $\varphi$ at these worlds, but may not be the minimum (maximum). A formula may therefore not be valid in such a model, but valid in all finite models where infima (suprema) and minima (maxima) coincide.
Theorem 7. Suppose that either (i) the universe of $A$ is $[0,1]$ or $G_\downarrow$, and the language contains $\rightarrow$, or (ii) the universe of $A$ is $G_\uparrow$ and the language contains $\rightarrow$ and $\Delta$. Then neither $K(A)$ nor $K(A)^C$ has the finite model property.

Proof. For (i), we follow [9] where it is shown that the following formula provides a counterexample to the finite model property of $GK$ and $GK^C$:

$$\Box \neg \neg p \rightarrow \neg \neg \Box p.$$ 

Just observe that the formula is valid in all finite $K(A)$-models, but not in the infinite $K(A)^C$-model $\langle \mathbb{N}^+, R, V \rangle$ where $Rmn = 1$ for all $m, n \in \mathbb{N}^+$ and $V(p, n) = \frac{1}{n}$ for all $n \in \mathbb{N}^+$. Hence neither $K(A)$ nor $K(A)^C$ has the finite model property.

Similarly, for (ii), the formula

$$\Delta \Diamond p \rightarrow \Diamond \Delta p$$

is valid in all finite $K(A)$-models, but not in the infinite $K(A)^C$-model $\langle \mathbb{N}^+, R, V \rangle$ where $Rmn = 1$ for all $m, n \in \mathbb{N}^+$ and $V(p, n) = \frac{n-1}{n}$ for all $n \in \mathbb{N}^+$. $\square$

Let us remark also that decidability and indeed PSPACE-completeness of validity in the box and diamond fragments of both $GK$ and $GK^C$ has been established in [26] using analytic Gentzen-style proof systems, but that decidability of validity in the full logics $GK$ and $GK^C$ has remained open.

3. A New Semantics for the Modal Operators

Consider again the failure of the finite model property for $GK^C$ established in the proof of Theorem 7. For a $GK^C$-model to render $\Box \neg \neg p \rightarrow \neg \neg \Box p$ invalid at a world $x$, there must be values of $p$ at worlds accessible to $x$ that form an infinite descending sequence tending to but never reaching $0$. This ensures that the infinite model falsifies the formula, but also that no particular world acts as a “witness” to the value of $\Box p$. Here, we redefine models to restrict the values at each world that can be taken by box-formulas and diamond-formulas. A formula such as $\Box p$ can then be “witnessed” at a world where the value of $p$ is merely “sufficiently close” to the value of $\Box p$.

To ensure that these redefined models accept the same valid formulas as the original models, we restrict our attention to order-based algebras where the order satisfies a certain homogeneity property. Recall that $R(A)$ and $L(A)$ are the sets of right and left accumulation points, respectively, of an order-based algebra $A$ in the usual topology inherited from $[0,1]$. Note also that by $(a,b)$, $[a,b)$, etc.
we denote here the intervals \((a, b) \cap A, [a, b) \cap A\), etc. in \(A\). We say that \(A\) is \textit{locally right homogeneous} if for any \(a \in R(A)\), there is a \(c \in A\) such that \(a < c\) and for any \(e \in (a, c)\), there is a complete order embedding \(h: [a, c) \rightarrow [a, e)\) such that \(h(a) = a\). In this case, \(c\) is called a \textit{witness} of right homogeneity at \(a\). Similarly, \(A\) is said to be \textit{locally left homogeneous} if for any \(b \in L(A)\), there is a \(d \in A\) such that \(d < b\) and for any \(f \in (d, b)\), there is a complete order embedding \(h: (d, b] \rightarrow (f, b]\) such that \(h(b) = b\). In this case, \(d\) is called a \textit{witness} of left homogeneity at \(b\). We will call \(A\) \textit{locally homogeneous} if it is both locally right homogeneous and locally left homogeneous.

Observe that if \(c \in A\) is a witness of right homogeneity at \(a\), then any \(e \in (a, c)\) will also be a witness of right homogeneity at \(a\). Hence \(c\) can be chosen sufficiently close to \(a\) so that \((a, c)\) is disjoint to any given finite set. A similar observation holds for witnesses of left homogeneity.

\textbf{Example 8.} Any finite \(A\) is trivially locally homogeneous. Also any \(A\) with \(A = [0, 1]\) is locally homogeneous: for \(a \in R(A) = [0, 1)\), choose any \(c > a\) to witness right homogeneity at \(a\), and similarly for \(b \in L(A) = (0, 1]\), choose any \(d < b\) to witness left homogeneity at \(b\). In the case of \(A = G^*_\uparrow\), \(L(A) = \emptyset\), \(R(A) = \{0\}\), and any \(c > 0\) witnesses right homogeneity at \(0\). Similarly, for \(A = G^*_\downarrow\), \(R(A) = \emptyset\), \(L(A) = \{1\}\), and any \(d < 1\) witnesses left homogeneity at \(1\). Moreover, infinitely many more non-isomorphic examples can be constructed using the fact that any ordered sum or lexicographical product of two locally homogeneous ordered sets is locally homogeneous.

Let us assume for the remainder of this section that \(A\) is a locally homogeneous order-based algebra. An \(FK(A)\)-model is a five-tuple \(\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle\) such that \(\langle W, R, V \rangle\) is a \(K(A)\)-model and \(T_\Box: W \rightarrow \mathcal{P}(A)\) and \(T_\Diamond: W \rightarrow \mathcal{P}(A)\) are functions satisfying for each \(x \in W\):

(i) \(C^A_L \subseteq T_\Box(x) \cap T_\Diamond(x)\),

(ii) \(T_\Box(x) = A \setminus \bigcup_{i \in I}(a_i, c_i)\) for some finite \(I \subseteq \mathbb{N}\) (possibly empty), where \(a_i \in R(A)\), \(c_i\) witnesses right homogeneity at \(a_i\), and the intervals \((a_i, c_i)\) are pairwise disjoint,

(iii) \(T_\Diamond(x) = A \setminus \bigcup_{j \in J}(d_j, b_j)\) for some finite \(J \subseteq \mathbb{N}\) (possibly empty), where \(b_j \in L(A)\), \(d_j\) witnesses left homogeneity at \(b_j\), and the intervals \((d_j, b_j)\) are pairwise disjoint.
The valuation \( V \) is extended to the mapping \( V : \text{Fm} \times W \) inductively as follows:

\[
V(*_n \varphi_1, \ldots, \varphi_n, x) = *_n(V(\varphi_1, x), \ldots, V(\varphi_n, x))
\]

for each \( n \)-ary operational symbol \( * \) of \( \mathcal{L} \), and

\[
\begin{align*}
V(\Box \varphi, x) &= \bigvee \{ r \in T_\Box(x) : r \leq \bigwedge \{ V(\varphi, y) : y \in W \} \} \\
V(\Diamond \varphi, x) &= \bigwedge \{ r \in T_\Diamond(x) : r \geq \bigvee \{ V(\varphi, y) : y \in W \} \}.
\end{align*}
\]

As before, an \( \text{FK}(A)^\subset \)-model satisfies the extra condition that \( \langle W, R \rangle \) is a crisp \( A \)-frame, and the conditions for \( \Box \) and \( \Diamond \) simplify to

\[
\begin{align*}
V(\Box \varphi, x) &= \bigvee \{ r \in T_\Box(x) : r \leq \bigwedge \{ V(\varphi, y) : Rxy \} \} \\
V(\Diamond \varphi, x) &= \bigwedge \{ r \in T_\Diamond(x) : r \geq \bigvee \{ V(\varphi, y) : Rxy \} \}.
\end{align*}
\]

A formula \( \varphi \in \text{Fm} \) is valid in \( \mathcal{M} \) if \( V(\varphi, x) = 1 \) for all \( x \in W \).

**Example 9.** Note that when \( A \) is finite, \( T_\Box(x) = T_\Diamond(x) = A \). For \( A = \{0, 1\} \), both \( T_\Box(x) \) and \( T_\Diamond(x) \) are obtained by removing finitely many arbitrary disjoint intervals \((a, b)\) not containing constants. For \( A = G_+ \), the only possibilities are \( T_\Box(x) = A \) and \( T_\Diamond(x) = A \) or \( T_\Box(x) = \{0, \frac{1}{n}, \frac{1}{n-1}, \ldots, 1\} \) for some \( n \in \mathbb{N}^+ \) respecting \( C_\mathcal{L} \subseteq T_\Box(x) \). The case of \( A = G_\bot \) is very similar.

It is worth pointing out that in every \( \text{FK}(A) \)-model \( \mathcal{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle \) and for any \( x \in W \), \( T_\Box(x) \) and \( T_\Diamond(x) \) will be complete subsets of \( A \). Hence, the suprema and infima defining \( V(\Box \varphi, x) \) and \( V(\Diamond \varphi, x) \) will actually be maxima and minima, and always \( V(\Box \varphi, x) \in T_\Box(x) \) and \( V(\Diamond \varphi, x) \in T_\Diamond(x) \).

We now extend some previously introduced notions to \( \text{FK}(A) \)-models. Given an \( \text{FK}(A) \)-model \( \mathcal{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle \), we call \( \mathcal{M}' = \langle W', R', V', T'_\Box, T'_\Diamond \rangle \) an \( \text{FK}(A) \)-submodel of \( \mathcal{M} \), written \( \mathcal{M}' \subseteq \mathcal{M} \), if \( W' \subseteq W \) and \( R', V', T'_\Box, T'_\Diamond \) are the restrictions to \( W' \) of \( R, V, T_\Box, T_\Diamond \) respectively. As before, given \( x \in W \), the \( \text{FK}(A) \)-submodel of \( \mathcal{M} \) generated by \( x \) is the smallest \( \text{FK}(A) \)-submodel \( \mathcal{M}' = \langle W', R', V', T'_\Box, T'_\Diamond \rangle \) of \( \mathcal{M} \) satisfying \( x \in W' \) and for all \( y \in W' \), \( z \in R^+[y] \) implies \( z \in W' \). Lemmas 2 and 3 then extend to \( \text{FK}(A) \)-models as follows with minimal changes in the proofs.

**Lemma 10.** Let \( \mathcal{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle \) be an \( \text{FK}(A) \)-model.

(a) Let \( \hat{\mathcal{M}} = \langle \hat{W}, \hat{R}, \hat{V}, \hat{T}_\Box, \hat{T}_\Diamond \rangle \) be a generated \( \text{FK}(A) \)-submodel of \( \mathcal{M} \). Then \( \hat{V}(\varphi, x) = V(\varphi, x) \) for all \( x \in \hat{W} \), and \( \varphi \in \text{Fm} \).
Given any $x \in W$ and $k \in \mathbb{N}$, there exists an FK($A$)-tree-model $\widehat{M} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_{\square}, \widehat{T}_{\Diamond} \rangle$ with root $\widehat{x}$ and $\text{hg}(\widehat{M}) \leq k$ such that $\widehat{V}(\varphi, \widehat{x}) = V(\varphi, x)$ for all $\varphi \in Fm$ with $\ell(\varphi) \leq k$, and if $M$ is an FK($A$)-C-model, then so is $\widehat{M}$.

Example 11. There are very simple finite FK($A$)$^C$-counter-models for the formula $\square \neg\neg p \rightarrow \neg\neg \square p$ when $A = [0, 1]$. For $M = \langle W, R, V, T_{\square}, T_{\Diamond} \rangle$ with $W = \{a\}$, $R_{aa} = 1$, $T_{\square}(a) = T_{\Diamond}(a) = C_L$, and $0 < V(p, a) < \min(C_L \setminus \{0\})$:

$$V(\square \neg\neg p, a) = \bigvee \{r \in C_L : r \leq \bigwedge \{V(\neg\neg p, y) : \text{Ray}\}\}$$
$$= \bigvee \{r \in C_L : r \leq V(\neg\neg p, a)\}$$
$$= \bigvee \{r \in C_L : r \leq 1\}$$
$$= 1$$

$$V(\neg\neg \square p, a) = \neg\neg \bigvee \{r \in C_L : r \leq \bigwedge \{V(p, y) : \text{Ray}\}\}$$
$$= \neg\neg \bigvee \{r \in C_L : r \leq V(p, a)\}$$
$$= \neg\neg 0$$
$$= 0$$

$$V(\square \neg\neg p \rightarrow \neg\neg \square p, a) = V(\square \neg\neg p, a) \rightarrow V(\neg\neg \square p, a)$$
$$= 1 \rightarrow 0$$
$$= 0.$$ 

The same formula fails in a similar finite FK($A$)$^C$-model when $A = G_\downarrow$, and $\Delta \Diamond p \rightarrow \Diamond \Delta p$ fails in a similar FK($A$)$^C$-model when $A = G_\uparrow$.

Indeed, as shown below, we can always “prune” (i.e., remove branches from) an FK($A$)-tree-model of finite height where $\varphi \in Fm$ is not valid in such a way that $\varphi$ is not valid in the new finite FK($A$)-model. It then follows from part (b) of Lemma 10 that FK($A$) and FK($A$)$^C$ have the finite model property.

Lemma 12. Let $\Sigma \subseteq Fm$ be a finite fragment. Then for any FK($A$)-tree-model $\mathfrak{M} = \langle W, R, V, T_{\square}, T_{\Diamond} \rangle$ of finite height with root $x$, there is a finite FK($A$)-tree-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V}, \widehat{T}_{\square}, \widehat{T}_{\Diamond} \rangle$ with $\langle \widehat{W}, \widehat{R}, \widehat{V} \rangle \subseteq \langle W, R, V \rangle$, root $x \in \widehat{W}$, and $|\widehat{W}| \leq |\Sigma|^{\text{hg}(\mathfrak{M})}$ such that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \Sigma$.

Proof. We prove the lemma by induction on $\text{hg}(\mathfrak{M})$. For the base case, $W = \{x\}$ and it suffices to define $\widehat{\mathfrak{M}} = \mathfrak{M}$. 
For the induction step $\text{hg}(\mathfrak{M}) = n + 1$, consider for each $y \in R^+[x]$, the submodel $\mathfrak{M}_y = \langle W_y, R_y, V_y, T_{\Sigma_y}, T_{\emptyset_y} \rangle$ of $\mathfrak{M}$ generated by $y$. Each $\mathfrak{M}_y$ is a $\text{FK}(A)$-tree-model of finite height with root $y$ and $\text{hg}(\mathfrak{M}_y) \leq n$. Hence, by the induction hypothesis, for each $y \in R^+[x]$, there is a finite $\text{FK}(A)$-tree-submodel $\hat{\mathfrak{M}}_y = \langle \hat{W}_y, \hat{R}_y, \hat{V}_y, \hat{T}_{\Sigma_y}, \hat{T}_{\emptyset_y} \rangle$ of $\mathfrak{M}$ with root $y \in \hat{W}_y$ such that $|\hat{W}_y| \leq |\Sigma|^n$ and for all $\varphi \in \Sigma$, $\hat{V}_y(\varphi, y) = V_y(\varphi, y)$ ($= V(\varphi, y)$ by Lemma 10(a)).

We choose a finite number of appropriate $y \in R^+[x]$ in order to build our finite $\text{FK}(A)$-submodel $\hat{\mathfrak{M}} = \langle \hat{W}, \hat{R}, \hat{V}, \hat{T}_{\square}, \hat{T}_{\emptyset} \rangle$ of $\mathfrak{M}$ as the “union” of these $\hat{\mathfrak{M}}_y$, connected by the root world $x \in \hat{W}$. First we define $\hat{T}_{\square}(x)$ and $\hat{T}_{\emptyset}(x)$.

Consider $T_{\square}(x) = A \setminus \bigcup_{i \in I}(a_i, c_i)$ for some finite $I \subseteq \mathbb{N}$ (possibly empty), where for all $i \in I$, $a_i \in R(A)$, $c_i$ witnesses right homogeneity at $a_i$, and the intervals $(a_i, c_i)$ are pairwise disjoint. Consider also the finite (possibly empty) set $(V[\Sigma_{\square}, x] \cap L(A)) \setminus \{a_i : i \in I\} = \{b_j : j \in J\}$ where $I \cap J = \emptyset$. For $j \in J$, choose a witness of right homogeneity $c_j$ at $a_j$ such that the intervals $(a_i, c_i)$ are pairwise disjoint, for all $i \in I \cup J$, and $(V[\Sigma_{\square}, x] \cup C_L) \cap (\bigcup_{i \in I \cup J}(a_i, c_i)) = \emptyset$.

We define $\hat{T}_{\square}(x) = A \setminus \bigcup_{i \in I \cup J}(a_i, c_i)$, satisfying conditions (i) and (ii) of the definition of an $\text{FK}(A)$-model by construction. Note also that $V[\Sigma_{\square}, x] \cup C_L \subseteq \hat{T}_{\square}(x) \subseteq T_{\square}(x)$.

Similarly, consider $T_{\emptyset}(x) = A \setminus \bigcup_{i \in I'}(d_i, b_i)$ for some finite $I' \subseteq \mathbb{N}$ (possibly empty), where for all $i \in I'$, $b_i \in L(A)$, $d_i$ witnesses left homogeneity at $b_i$, and the intervals $(d_i, b_i)$ are pairwise disjoint. Consider also the finite (possibly empty) set $(V[\Sigma_{\emptyset}, x] \cap L(A)) \setminus \{b_i : i \in I'\} = \{d_j : j \in J'\}$. For $j \in J'$, choose a witness of left homogeneity $c_j$ at $b_j$ such that the intervals $(d_i, b_i)$ are pairwise disjoint for all $i \in I' \cup J'$, and $(V[\Sigma_{\emptyset}, x] \cup C_L) \cap (\bigcup_{i \in I' \cup J'}(d_i, b_i)) = \emptyset$. We define $\hat{T}_{\emptyset}(x) = A \setminus \bigcup_{i \in I' \cup J'}(d_i, b_i)$, satisfying conditions (i) and (iii) of the definition of an $\text{FK}(A)$-model by construction. Note also that $V[\Sigma_{\emptyset}, x] \cup C_L \subseteq \hat{T}_{\emptyset}(x) \subseteq T_{\emptyset}(x)$.

Consider now $\varphi = \Box \psi \in \Sigma_{\square}$ and let $a = V(\Box \psi, x) \in \hat{T}_{\square}(x)$. If $a \notin R(A)$, choose $y_{\varphi} \in R^+[x]$ such that $a = Rxy_{\varphi} \rightarrow V(\psi, y_{\varphi})$. If $a \in R(A)$, there is an $i \in I \cup J$, such that $a = a_i$, and we choose $y_{\varphi} \in R^+[x]$ such that $Rxy_{\varphi} \rightarrow V(\psi, y_{\varphi}) \in [a_i, c_i]$. Similarly, for each $\varphi = \Diamond \psi \in \Sigma_{\emptyset}$, let $b = V(\Diamond \psi, x) \in \hat{T}_{\emptyset}(x)$. If $b \notin L(A)$, choose $y_{\varphi} \in R^+[x]$ such that $b = Rxy_{\varphi} \land V(\psi, y_{\varphi})$. If $b \in L(A)$, there is an $i \in I' \cup J'$, such that $b = b_i$ and we choose $y_{\varphi} \in R^+[x]$ such that $Rxy_{\varphi} \land V(\psi, y_{\varphi}) \in [d_i, b_i]$.

Now let $Y = \{y_{\varphi} \in R^+[x] : \varphi \in \Sigma_{\square} \cup \Sigma_{\emptyset}\}$, noting that $|Y| \leq |\Sigma_{\square} \cup \Sigma_{\emptyset}| < |\Sigma|$.
We define $\hat{\mathcal{M}} = \langle \hat{W}, \hat{R}, \hat{V}, \hat{T}_\Box, \hat{T}_\Diamond \rangle$ where

$$\hat{W} = \{x\} \cup \bigcup_{y \in Y} \hat{W}_y,$$

and $\hat{R}$ and $\hat{V}$ are $R$ and $V$, respectively, restricted to $\hat{W}$. $\hat{T}_\Box$ and $\hat{T}_\Diamond$ are defined as $T_\Box$, and $T_\Diamond$, respectively, on $\hat{W} \setminus \{x\}$ and as defined above for $x$.

Observe that $\hat{\mathcal{M}}$ is an $\hat{\mathcal{N}}$-tree-submodel of $\mathcal{M}$ with root $x \in \hat{W}$ and $|\hat{W}| \leq |Y||\Sigma|^n + 1 < |\Sigma||\Sigma|^n = |\Sigma|^{\mathcal{M}(\mathcal{M})}$. Moreover, for each $y \in Y$, $\hat{\mathcal{M}}_y$ is an $\hat{\mathcal{N}}(\mathcal{A})$-submodel of $\hat{\mathcal{M}}$ generated by $y$. Hence, by Lemma 10(a) and the induction hypothesis, for all $\phi \in \Sigma$,

$$\hat{V}(\phi, y) = \hat{V}_y(\phi, y) = V_y(\phi, y) = V(\phi, y). \quad (15)$$

We show now that $\hat{V}(\phi, x) = V(\phi, x)$ for all $\phi \in \Sigma$, proceeding by induction on $\ell(\phi)$. The base case follows directly from the definition of $\hat{V}$. For the inductive step, the non-modal cases follow directly using the induction hypothesis. For $\phi = \Box \psi$, there are two cases. Suppose first that $V(\Box \psi, x) = a \notin R(\mathcal{A})$ and recall that

$$V(\Box \psi, x) = \bigvee \{r \in T_\Box(x) : r \leq \bigwedge \{Rxy \rightarrow V(\psi, y) : y \in W\}\} = a.$$

This implies that $\hat{R}xy \rightarrow \hat{V}(\psi, y) \geq a$ for all $y \in Y \subseteq R^+[x]$. Hence, by (15), $\hat{R}xy \rightarrow \hat{V}(\psi, y) \geq a$ for all $y \in Y = \hat{R}^+[x]$. Moreover, $\hat{R}xy, \phi \rightarrow \hat{V}(\psi, y, \phi) = a$ and hence, because $a \in V[\Sigma, x] \subseteq \hat{T}_\Box(x)$,

$$\hat{V}(\Box \psi, x) = \bigvee \{r \in \hat{T}_\Box(x) : r \leq \bigwedge \{\hat{R}xy \rightarrow \hat{V}(\psi, y) : y \in \hat{W}\}\} = a.$$

For the second case, suppose that $V(\Box \psi, x) = a \in R(\mathcal{A})$. Then $a = a_i$ for some $i \in I \cup J$ and, because $\hat{T}_\Box(x) \subseteq T_\Box(x)$,

$$a = a_i \leq \bigwedge \{Rxy \rightarrow V(\psi, y) : y \in W\}.$$

But $\hat{W} \subseteq W$ and $\hat{R}xy \rightarrow \hat{V}(\psi, y) = Rxy \rightarrow V(\psi, y)$ for each $y \in \hat{W}$, so also

$$a_i \leq \bigwedge \{Rxy \rightarrow V(\psi, y) : y \in W\} \leq \bigwedge \{\hat{R}xy \rightarrow \hat{V}(\psi, y) : y \in \hat{W}\}.$$

By the choice of $y, \phi \in \hat{W}$,

$$\hat{R}xy, \phi \rightarrow \hat{V}(\psi, y, \phi) = Rxy, \phi \rightarrow V(\psi, y, \phi) < c_i.$$
Hence \( a_i \leq \bigwedge \{ \hat{R}xy \to \hat{V}(\psi, y) : y \in \hat{W} \} < c_i \) and
\[
\hat{V}(\Box \psi, x) = \bigvee \{ r \in \hat{T}_{\Box}(x) : r \leq \bigwedge \{ \hat{R}xy \to \hat{V}(\psi, y) : y \in \hat{W} \} \} = a_i = a.
\]
The case where \( \varphi = \Diamond \psi \) is very similar. \( \square \)

**Corollary 13.** \( \text{FK}(A) \) and \( \text{FK}(A)^C \) have the finite model property.

### 4. Equivalence of the Semantics

Let us assume again that \( A \) is a locally homogeneous order-based algebra. We devote this section to establishing that a formula is valid in \( K(A) \) or \( K(A)^C \) if and only if it is valid in \( \text{FK}(A) \) or \( \text{FK}(A)^C \), respectively. Observe first that any \( K(A) \)-model can be extended to an \( \text{FK}(A) \)-model with the same valid formulas simply by defining \( T_{\Box} \) and \( T_{\Diamond} \) to be constantly \( A \). Hence any \( \text{FK}(A) \)-valid formula is also \( K(A) \)-valid. We therefore turn our attention to the other (much harder) direction: proving that any \( K(A) \)-valid formula is also \( \text{FK}(A) \)-valid.

The main ingredient of the proof (see Lemma 16) is the construction of a \( K(A) \)-tree-model taking the same values for formulas at its root as a given \( \text{FK}(A) \)-tree-model. Note that the original \( \text{FK}(A) \)-tree-model without the functions \( T_{\Box} \) and \( T_{\Diamond} \) cannot play this role in general; in \([0, 1]\), for example, the infimum or supremum required for calculating the value of a box-formula or diamond-formula at the root \( x \) might not be in the set \( T_{\Box}(x) \cup T_{\Diamond}(x) \). This problem is resolved by taking infinitely many copies of an inductively defined \( K(A) \)-model in such a way that certain parts of the intervals between members of \( T_{\Box}(x) \) and \( T_{\Diamond}(x) \) are “squeezed” closer to either their lower or upper bounds. The obtained infima and suprema will then coincide with the next smaller or larger member of \( T_{\Box}(x) \) and \( T_{\Diamond}(x) \): that is, the required values of the formulas at \( x \) in the original \( \text{FK}(A) \)-tree-model. The following example illustrates this idea for the relatively simple case where \( A = G \).

**Example 14.** Consider the \( \text{FK}(G) \)-tree-model \( \mathcal{M} = \langle W, R, V, T_{\Box}, T_{\Diamond} \rangle \) with \( W = \{ x, y \} \), \( R = \{(x, y)\} \), and \( T_{\Box}(x) = [0, 1] \setminus (0.2, 0.8) \). Note that \( 0.2 \in R(G) \) and that \( 0.8 \) witnesses right homogeneity at \( 0.2 \). Suppose that \( V(p, y) = 0.6 \), so that
\[
V(\Box p, x) = \bigvee \{ r \in T_{\Box}(x) : r \leq \bigwedge \{ V(p, y) : Rxy \} \}
= \bigvee \{ r \in [0, 1] \setminus (0.2, 0.8) : r \leq 0.6 \}
= 0.2.
\]
For each $k \geq 2$, we then consider $\mathcal{M}_k = \langle W_k, R_k, V_k \rangle$ with $W_k = \{y_k\}$, $R_k = \emptyset$, and $V_k(p, y_k) = h_k(V(p, y))$, for some $\{0, 1\}$-order embedding $h_k : [0, 1] \to [0, 1]$, satisfying for each $k \geq 2$,

$$h_k[[0.2, 0.8]] = [0.2, 0.2 + \frac{1}{k}).$$

Defining the $\mathcal{K}(G)^G$-tree-model $\widehat{\mathcal{M}} = \langle \widehat{W}, \widehat{R}, \widehat{V} \rangle$, with $\widehat{W} = \{x\} \cup \{y_k : k \geq 2\}$, $\widehat{R} = \{(x, y_k) : k \geq 2\}$, and $\widehat{V}(p, y_k) = V_k(p, y_k)$, we obtain (see Figure 1):

$$\widehat{V}(\square p, x) = \bigwedge \{\widehat{V}(p, y_k) : \widehat{R}xy_k\}$$

$$= 0.2$$

$$= V(\square p, x).$$

A central tool in the proof of Lemma 16 is the following result which allows the "squeezing" of $\mathcal{K}(A)$-models so that the values of formulas are arbitrarily close to certain points (as in Example 14). Intuitively, in the proof of Lemma 16, the set $B$ below will be the set of all values at the root world $x$ of all box-formulas and diamond-formulas in some fragment $\Sigma$. In (a) below, the value $t$, the upper endpoint of the squeezed interval, will then be chosen in $A \setminus B \cap L(A)$ in order to ensure that all the suprema in $B$ (relevant for determining the values of diamond-formulas in $\Sigma$) are preserved by the squeezing. The values $a$ and $c$ will denote the endpoints of the removed interval and $s$ will be the relevant value that we want to squeeze closer and closer towards $a$. Note that $u \in [0, 1]$ can be any value as close to $a$ as needed (e.g., $u = a + \frac{1}{k}$ for any $k \in \mathbb{N}$) so as to squeeze the interval $[a, t)$ into $[a, u)$ by the $B$-complete order embedding $h$, with the intention that $s \in [a, t)$ and $h(s) \in [a, u)$. For (b), the ideas are very similar.

**Lemma 15.** Let $B \subseteq A$ be countable.

(a) Given $a \in R(A)$, some witness $c > a$ of right homogeneity at $a$, and an $s \in [a, c)$, there is a $t \in (s, c]$ such that $t \notin B \cap L(A)$. Moreover, for all $u \in (a, t]$, there is a $B$-complete order embedding $h : A \to A$ such that

$$h[[a, t]] \subseteq [a, u), \quad \text{and} \quad h|_{A \setminus (a, t)} = \text{id}_A.$$
\[ V(p, x) = 0.2 \]

\[ V(p, y) \]

\[ V_k(p, y_k) \]

\[ \hat{V}(\square p, x) = 0.2 \]

\[ \{ \hat{V}(p, y_k) : k \geq 2 \} \]

**Figure 1:** Squeezing models
(b) Given $b \in L(A)$, some witness $d < b$ of left homogeneity at $b$, and an $s \in (d, b]$, there is $t \in [d, s)$ such that $t \notin B \cap R(A)$. Moreover, for all $u \in [t, b)$, there is a $B$-complete order embedding $h : A \to A$ such that

$$h[(t, b)] \subseteq (u, b] \quad \text{and} \quad h|_{A \setminus (t, b)} = \text{id}_A.$$  

**Proof.** For (a), let $B \subseteq A$ be countable and consider $a \in R(A)$, a witness $c$ of right homogeneity at $a$, and $s \in [a, c)$. We first prove that there is a $t \in (s, c]$ which is either in $A \setminus L(A)$ or in $A \setminus B$. If $c \notin L(A)$, choose $t = c$. If $c \in L(A)$, then $[s, c]$ is infinite. Recall that every non-empty perfect set of real numbers (closed and containing no isolated points) is uncountable. Hence if $[s, c]$ is countable, there must be an isolated point $t \in (s, c]$ such that $t \notin L(A)$. If $[s, c]$ is uncountable, then there is a $t \in (s, c] \setminus B$, as $B$ is countable. Either way, there is a $t \in (s, c]$ such that $t \notin B \cap L(A)$.

Now we define the embedding. Because $t \leq c$ also witnesses right homogeneity at $a$, there is for each $u \in (a, t]$ a complete order embedding $g : [a, t) \to [a, u)$ with $g(a) = a$. Define $h$ as $g$ on $[a, t)$ and as the identity on $A \setminus (a, t)$. Then all arbitrary meets and joins in $A$ are preserved except in the case where $t$ is a join of elements in $[a, t)$ and so $t \in L(A)$. But in this case $t \notin B$. Hence (a) holds. For (b), we use a very similar argument. \hfill \Box

**Lemma 16.** Let $\Sigma$ be a finite fragment and let $M = \langle W, R, V, T_\Box, T_\Diamond \rangle$ be a finite FK($A$)-tree-model with root $x$. Then there is a countable $K(A)$-tree-model $\hat{M} = \langle \hat{W}, \hat{R}, \hat{V} \rangle$ with root $\hat{x}$ such that $\hat{V}(\varphi, \hat{x}) = V(\varphi, x)$ for all $\varphi \in \Sigma$. Moreover, if $M$ is crisp, then so is $\hat{M}$.

**Proof.** The lemma is proved by induction on $\text{hg}(M)$. The base case is immediate, fixing $\hat{M} = \langle \hat{W}, \hat{R}, \hat{V} \rangle$ with $\hat{W} = W = \{x\}$, $\hat{R} = R$, and $\hat{V} = V$. For the induction step, given $y \in R^+[x]$, let $M_y = \langle W_y, R_y, V_y, T_{\Box y}, T_{\Diamond y} \rangle$ be the submodel of $M$ generated by $y$. Then $M_y$ is a finite FK($A$)-tree-model with root $y$, $\text{hg}(M_y) < \text{hg}(M)$, and, by Lemma 10(a), $V_y(\varphi, z) = V(\varphi, z)$ for all $z \in W_y$ and $\varphi \in \Sigma$. So, by the induction hypothesis, there is a countable $K(A)$-tree-model $\hat{M}_y = \langle \hat{W}_y, \hat{R}_y, \hat{V}_y \rangle$ (crisp if $M$ is crisp) with root $\hat{y}$ such that $\hat{V}_y(\varphi, \hat{y}) = V_y(\varphi, y) = V(\varphi, y)$ for all $\varphi \in \Sigma$.

For each $\varphi \in \Sigma_\Box \cup \Sigma_\Diamond$, we will choose a world $y_\varphi \in R^+[x]$ as described below and then, using Lemma 15, define for each $k \in \mathbb{N}$ a copy of the $K(A)$-tree-model $\hat{M}_y$, denoted $\hat{M}_y^k$. Suppose that $\varphi = \Box \psi \in \Sigma_\Box$. Consider $T_{\Box}(x) = A \setminus \bigcup_{i \in I} (a_i, c_i)$ for some finite $I \subseteq \mathbb{N}$ (possibly empty), where for all $i \in I$, \hfill \Box
\(a_i \in R(A),\ c_i\) witnesses right homogeneity at \(a_i\), and the intervals \((a_i, c_i)\) are pairwise disjoint. There are two cases.

(i) Suppose that \(V(\square \psi, x) = a_i\) for some \(i \in I\). Recalling that

\[a_i = V(\square \psi, x) = \bigvee \{r \in T_\square(x) : r \leq \bigwedge \{Rx y \to V(\psi, y) : y \in W\}\},\]

there must be a world \(y_\varphi \in R^+[x]\) such that

\[Rx y_\varphi \to V(\psi, y_\varphi) \in [a_i, c_i].\]

We fix \(B = \widehat{\psi}_y [\Sigma_\square \cup \Sigma_\Box, \widehat{W}_{y_\varphi}]\). Using Lemma 15, for some \(t\) satisfying

\[a_i \leq s = Rx y_\varphi \to V(\psi, y_\varphi) < t \leq c_i,\]

there exists for each \(k \in \mathbb{N}^+\), a \(B\)-complete order embedding \(h_k : A \to A\) mapping \([a_i, t]\) into \([a_i, a_i + \frac{1}{k})\), and \(h_k|_{A\setminus(a_i, t)} = \text{id}_A\). Clearly, this implies that for all \(k \in \mathbb{N}^+,\ h_k\) is a \(\widehat{\psi}_y [\Sigma_\square \cup \Sigma_\Box, \widehat{W}_{y_\varphi}]\)-complete \(C_L\)-order embedding. We then define the copy \(\widehat{\text{M}}^k_\varphi = (\widehat{W}^k_\varphi, \hat{\psi}^k_\varphi, \hat{\phi}^k_\varphi)\) of \(\text{M}_{y_\varphi}\) as follows:

- \(\widehat{W}^k_\varphi\) is a copy of \(\widehat{W}_{y_\varphi}\), denoting the copy of \(\widehat{x}_{y_\varphi} \in \widehat{W}_{y_\varphi}\) by \(\hat{x}^k_\varphi\)
- \(\hat{\psi}^k_\varphi, \hat{x}^k_\varphi, \hat{z}^k_\varphi = h_k(\hat{\psi}_y, \hat{x}_y, \hat{z}_y)\) for \(\hat{x}_y, \hat{z}_y \in \widehat{W}_{y_\varphi}\)
- \(\hat{\psi}^k_\varphi(p, \hat{x}^k_\varphi) = h_k(\hat{\psi}_{y_\varphi}(p, \hat{x}_{y_\varphi}))\) for \(\hat{x}_{y_\varphi} \in \widehat{W}_{y_\varphi}\).

Because \(h_k\) is a \(\widehat{\psi}_y [\Sigma_\square \cup \Sigma_\Box, \widehat{W}_{y_\varphi}]\)-complete \(C_L\)-order embedding, by Lemma 1,

\(\hat{\psi}^k_\varphi(\chi, \hat{\theta}^k_\varphi) = h_k(\hat{\psi}_{y_\varphi}(\chi, \hat{\theta}_{y_\varphi}))\) for all \(\chi \in \Sigma\). By the induction hypothesis,

\[h_k(Rx y_\varphi) \to \hat{\psi}^k_\varphi(\varphi, \hat{\theta}^k_\varphi) = h_k(Rx y_\varphi) \to h_k(\hat{\psi}_{y_\varphi}(\psi, \hat{\theta}_{y_\varphi})) = h_k(Rx y_\varphi) \to \hat{\psi}_{y_\varphi}(\psi, \hat{\theta}_{y_\varphi}) = h_k(Rx y_\varphi) \to V(\psi, y_\varphi) = h_k(s) \in [a_i, a_i + \frac{1}{k}).\]

(ii) Suppose that \(V(\square \psi, x) \neq a_i\) for all \(i \in I\). In this case, \(V(\square \psi, x) = \bigwedge \{Rx y \to V(\psi, y) : y \in W\}\) and because \(W\) is finite, there is a \(y_\varphi \in W\), such that, by the induction hypothesis,

\[V(\square \psi, x) = Rx y_\varphi \to V(\psi, y_\varphi) = Rx y_\varphi \to \hat{\psi}_{y_\varphi}(\psi, y_\varphi).\]
In this case, let \( h_k \) be the identity function on \( A \) and \( \widehat{\mathcal{M}}^k_\varphi = (\hat{\mathcal{W}}^k_\varphi, \hat{\mathcal{R}}^k_\varphi, \hat{\mathcal{V}}^k_\varphi) = \widehat{\mathcal{M}}_{y_\varphi} \).

Similarly, when \( \varphi = \diamond \psi \in \Sigma_0 \), we obtain for each \( k \in \mathbb{N}^+ \), a \( \mathcal{K}(A) \)-tree-model \( \widehat{\mathcal{M}}^k_\varphi \) as a copy of \( \widehat{\mathcal{M}}_{y_\varphi} \).

We now define the \( \mathcal{K}(A) \)-tree-model \( \hat{\mathcal{M}} = (\hat{\mathcal{W}}, \hat{\mathcal{R}}, \hat{\mathcal{V}}) \) by

\[
\hat{\mathcal{W}} = \{ \hat{x} \} \cup \bigcup_{\varphi \in \Sigma_0} \bigcup_{k \in \mathbb{N}^+} \hat{\mathcal{W}}^k_\varphi
\]

\[
\hat{\mathcal{R}}_{\varphi} w z = \begin{cases} \hat{\mathcal{R}}^k_\varphi w z & \text{if } w, z \in \hat{\mathcal{W}}^k_\varphi \text{ for some } \varphi \in \Sigma_0 \cup \Sigma_0, k \in \mathbb{N}^+ \\ h_k (R_{xy}) & \text{if } w = \hat{x}, z = \hat{\gamma}^k_\varphi \in \hat{\mathcal{W}}^k_\varphi \text{ for } \varphi \in \Sigma_0 \cup \Sigma_0, k \in \mathbb{N}^+ \\ 0 & \text{otherwise} \end{cases}
\]

\[
\hat{\mathcal{V}}(p, z) = \begin{cases} \hat{\mathcal{V}}^k(p, z) & \text{if } z \in \hat{\mathcal{W}}^k_\varphi \text{ for some } \varphi \in \Sigma_0 \cup \Sigma_0, k \in \mathbb{N}^+ \\ V(p, x) & \text{if } z = \hat{x} \end{cases}
\]

If \( \hat{\mathcal{M}} \) is crisp, then for all \( \varphi \in \Sigma_0 \cup \Sigma_0 \), \( \widehat{\mathcal{M}}_{y_\varphi} \) is crisp and so also are \( \widehat{\mathcal{M}}^k_\varphi \) for all \( k \in \mathbb{N}^+ \). Hence, by construction, \( \hat{\mathcal{M}} \) is crisp. Moreover, as there are only finitely many different countable \( \widehat{\mathcal{M}}_{y_\varphi} \), and we only take countably many copies of each one, \( \hat{\mathcal{M}} \) is also countable.

Observe now that for each \( \hat{\gamma}^k_\varphi \in \hat{\mathcal{R}}^+ [\hat{x}] \), we have that \( \widehat{\mathcal{M}}^k_\varphi \) is the submodel of \( \hat{\mathcal{M}} \) generated by \( \hat{\gamma}^k_\varphi \). Hence, by Lemma 2, for all \( \chi \in \Sigma \) and \( \hat{\gamma}^k_\varphi \in \hat{\mathcal{R}}^+ [\hat{x}] \),

\[
\hat{\mathcal{V}}(\chi, \hat{\gamma}^k_\varphi) = \hat{\mathcal{V}}^k(\chi, \hat{\gamma}^k_\varphi) = h_k (\hat{\mathcal{V}}_{y_\varphi}(\chi, \hat{\gamma}^k_\varphi)) = h_k (V_{y_\varphi}(\chi, y_\varphi)) = h_k (V(\chi, y_\varphi)).
\]

Finally, we prove that \( \hat{\mathcal{V}}(\chi, \hat{x}) = V(\chi, x) \) for all \( \chi \in \Sigma \), proceeding by induction on \( \ell(\chi) \). The base case follows directly from the definition of \( \hat{\mathcal{V}} \). For the induction step, the cases for the non-modal connectives follow easily using the induction hypothesis. Let us just consider the case \( \chi = \varphi = \square \psi \) (a formula in \( \Sigma_0 \)), the case \( \chi = \diamond \psi \) being very similar. There are two cases.

(i) Suppose that \( V(\square \psi, x) = a_i \) for some \( i \in I \). Then for all \( z \in W \), we have \( Rxz \rightarrow V(\psi, z) \geq a_i \). Note that it is not possible that for some \( a \in A \) and some \( h_k \) defined above: \( a \geq a_i \) and \( h_k(a) < a_i \). So by construction, for all \( \hat{z} \in \hat{W} \),

\[
\hat{R} \hat{z} \hat{x} \hat{z} \rightarrow \hat{V}(\psi, \hat{z}) \geq a_i.
\]

Moreover, for each \( y_\varphi \in W \),

\[
Rxy_\varphi \rightarrow V(\psi, y_\varphi) \in [a_i, c_i),
\]

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and by the induction hypothesis and $(*),$
\[
a_i \leq \bigwedge \{ \overline{R} \overline{x} \overline{z} \rightarrow \widehat{V}(\psi, \overline{z}) : \overline{z} \in \widehat{W} \} \\
\leq \bigwedge \{ \overline{R} \overline{x} \overline{y}_{\phi}^k \rightarrow \widehat{V}(\psi, \overline{y}_{\phi}^k) : k \in \mathbb{N}^+ \} \\
= \bigwedge \{ h_k(Rxy_{\varphi}) \rightarrow \widehat{V}_{\psi}^k(\psi, \overline{y}_{\phi}^k) : k \in \mathbb{N}^+ \} \\
= \bigwedge \{ h_k(Rxy_{\varphi} \rightarrow V(\psi, y_{\varphi})) : k \in \mathbb{N}^+ \} \\
\leq \bigwedge \{ a_i + \frac{1}{k} : k \in \mathbb{N}^+ \} \\
= a_i.
\]
So \(\widehat{V}(\square \psi, \overline{x}) = \bigwedge \{ \overline{R} \overline{x} \overline{z} \rightarrow \widehat{V}(\psi, \overline{z}) : \overline{z} \in \widehat{W} \} = a_i = V(\square \psi, x)\) as required.

(ii) Suppose that \(V(\square \psi, x) \neq a_i\) for all \(i \in I\). Again, we have that for all \(z \in W, \overline{R} \overline{x} \overline{z} \rightarrow V(\psi, z) \geq V(\square \psi, x)\). So by construction, for all \(\overline{z} \in \widehat{W},\)
\[
\overline{R} \overline{x} \overline{z} \rightarrow \widehat{V}(\psi, \overline{z}) \geq V(\square \psi, x).
\]
Moreover, as in (ii) above, because \(W\) is finite, there is a \(y_{\varphi} \in W\) such that
\[
Rxy_{\varphi} \rightarrow V(\psi, y_{\varphi}) = V(\square \psi, x).
\]
Using the induction hypothesis and the fact that \(h_k\) is the identity on \(\overline{T}(\square \psi, x)\),
\[
\widehat{V}(\square \psi, \overline{x}) = \bigwedge \{ \overline{R} \overline{x} \overline{z} \rightarrow \widehat{V}(\psi, \overline{z}) : \overline{z} \in \widehat{W} \} \\
= \bigwedge \{ \overline{R} \overline{x} \overline{y}_{\phi}^k \rightarrow \widehat{V}(\psi, \overline{y}_{\phi}^k) : k \in \mathbb{N}^+ \} \\
= \bigwedge \{ h_k(Rxy_{\varphi}) \rightarrow \widehat{V}_{\psi}^k(\psi, \overline{y}_{\phi}^k) : k \in \mathbb{N}^+ \} \\
= \bigwedge \{ h_k(Rxy_{\varphi} \rightarrow V(\psi, y_{\varphi})) : k \in \mathbb{N}^+ \} \\
= Rxy_{\varphi} \rightarrow V(\psi, y_{\varphi}) \\
= V(\square \psi, y_{\varphi})
\]
So \(\widehat{V}(\square \psi, \overline{x}) = V(\square \psi, x)\) as required. \(\Box\)

We obtain the following equivalence result.

**Theorem 17.**

(a) \(\models_{K(A)} \varphi \) if and only if \(\models_{FK(A)} \varphi\).
(b) \( \models_{K(A)^C} \varphi \) if and only if \( \models_{FK(A)^C} \varphi \).

**Proof.** For (a), the right-to-left direction is immediate using the fact that every \( K(A) \)-tree-model can be extended to an \( FK(A) \)-tree-model with the same valid formulas by setting \( T_\square \) and \( T_\lozenge \) to be constantly \( A \). Suppose now that \( \not\models_{FK(A)^C} \varphi \).

By Lemmas 10 and 12, there is a finite \( FK(A) \)-tree-model \( M = \langle W, R, V, T_\square, T_\lozenge \rangle \) with root \( x \) such that \( V(\varphi, x) < 1 \). By Lemma 16, we obtain a \( K(A) \)-tree-model \( \hat{M} = \langle \hat{W}, \hat{R}, \hat{V} \rangle \) with root \( \hat{x} \) such that \( \hat{V}(\varphi, \hat{x}) = V(\varphi, x) < 1 \). So \( \not\models_{K(A)} \varphi \).

The proof of (b) is very similar using the fact that Lemmas 10, 12, and 16 preserve crisp models. \( \square \)

5. Decidability and Complexity

Let us assume again that \( A \) is a locally homogeneous order-based algebra. In this section, we will use the finite model property of \( FK(A) \) and \( FK(A)^C \) to obtain decidability and complexity results for \( K(A) \) and \( K(A)^C \) in various cases.

We prove, in particular, that the Gödel modal logics \( GK \) and \( GK^C \) (i.e., where \( A \) is \( G \)) are both \( PSPACE \)-complete and that the same is true for the cases where \( A \) is \( G_\downarrow \) or \( G_\uparrow \). These and other results in this section contrast with the fact that no first-order Gödel logic based on a countably infinite set of truth values is recursively axiomatizable [1].

For simplicity of exposition, we will assume that the only constants are \( \bar{0} \) and \( \bar{I} \). To explain the ideas involved in the proofs, consider \( \varphi \in \text{Fm} \) and \( n = \ell(\varphi) + |C_L| = \ell(\varphi) + 2 \). To check whether \( \varphi \) is not \( K(A) \)-valid, it suffices, by Lemmas 10, 12, and 16, to find a finite \( FK(A) \)-tree-model \( M = \langle W, R, V, T_\square, T_\lozenge \rangle \) of height \( \leq n \) with root \( x \) and \( |W| \leq |\Sigma(\varphi)|^{\ell(\varphi)} \leq n^n \) such that \( V(\varphi, x) < 1 \).

If \( A \) is infinite, then \( T_\square(y) \) and \( T_\lozenge(y) \) may also be infinite, and hence \( M \) may not be a computational object. We therefore introduce a modified version of \( M \):

\[ M^* = \langle W, R, V, \{\Phi(y)\}_{y \in W}, \{\Psi(y)\}_{y \in W} \rangle, \]

where for each \( y \in W \), \( \Phi(y) \subseteq A^2 \) is the set of ordered pairs for which \( T_\square(y) = A \setminus \bigcup_{(r,s) \in \Phi(y)} (r, s) \), and \( \Psi(y) \subseteq A^2 \) is the set of ordered pairs defining \( T_\lozenge(y) \). From the proof of Lemma 12 applied to a \( K(A) \)-model, we may assume \( |\Phi(y)|, |\Psi(y)| \leq |\Sigma(\varphi)| \leq n \) for all \( y \in W \), because the left endpoints of the intervals utilized in the proof to define \( \hat{T}_\square(x) \) in the finite \( FK(A) \)-tree-model belong to \( V[\Sigma(\varphi)_\square, x] \), and similarly for \( \hat{T}_\lozenge(x) \). Let us define inductively in \( M^* \), for all
$y \in W$ and all $\psi \in F_m$,

\[
V(\Box \psi, y) = \begin{cases} 
  r & \text{if } \bigwedge_{y \in W}(Rxy \to V(\psi, y)) \in (r, s) \\
  \bigwedge_{y \in W}(Rxy \to V(\psi, y)) & \text{otherwise},
\end{cases}
\]

\[
V(\Diamond \psi, y) = \begin{cases} 
  s & \text{if } \bigvee_{y \in W}(Rxy \land V(\psi, y)) \in (r, s) \\
  \bigvee_{y \in W}(Rxy \land V(\psi, y)) & \text{otherwise}.
\end{cases}
\]

Then $\mathcal{M}^*$ and $\mathcal{M}$ assign the same values to any formula at any world. Moreover, for $\chi \in \Sigma(\varphi)$, the computation of $V(\chi, y)$ in $\mathcal{M}^*$ involves only the set of values $N = V[\Sigma(\varphi), W] \cup \{Ryz : y, z \in W\} \cup \{r, s : \langle r, s \rangle \in \Phi(\cdot) \cup \Psi(\cdot), y \in W\}$.

Note that $|N| \leq 4n^{2n} = e_n$. Therefore, we may assume that $R$ and $V$ take values in the fixed set $A(e_n)$, where for any $m \in \mathbb{N}^+$,

\[A(m) = \{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1\},\]

and also that $W$ is $W_n \subseteq \{0, 1, \ldots, n^n\}$, $x = 0$, and $\Phi(i), \Psi(i) \subseteq A(e_n)^2$ for all $i \leq e_n$. This yields a finite $A(e_n)$-valued structure:

\[\mathcal{M}^*(e_n) = \langle W_n, R, V, \{\Phi(i)\}_{i \in W_n}, \{\Psi(i)\}_{i \in W_n}\rangle,
\]

where $\langle W_n, R^+ \rangle$ is a tree of height and branching $\leq n$ with root 0, and the sets $\Phi(i), \Psi(i)$, for all $i \in W_n$, determine the endpoints of a family of disjoint open intervals in $A(e_n)$. We will call this kind of structure an $\text{FK}(e_n)$-tree-model. In order to recover the connection with the original $\text{FK}(A)$-model, we introduce the following convenient notion.

A finite system is a triple $A(m) = \langle A(m), \Phi, \Psi \rangle$ where $\Phi, \Psi \subseteq A(m)^2$. We call $A(m)$ consistent with $A$ if for some order-preserving embedding $h: A(m) \to A$, satisfying $h(0) = 0$ and $h(1) = 1$,

- $h(c)$ witnesses right homogeneity at $h(a) \in R(A)$ for all $\langle a, c \rangle \in \Phi$,
- $h(d)$ witnesses left homogeneity at $h(b) \in L(A)$ for all $\langle d, b \rangle \in \Psi$.

Then we obtain from the previous discussion:
Theorem 18. The validity problems of $K(A)$ and $K(A)^C$ are decidable if the problem of consistency of finite systems $A(m)$ with $A$ is decidable. In fact, the first problems are co-NEXPTIME reducible (in the length of the formula).

Proof. We have seen that $\varphi \in Fm$ with $n = \ell(\varphi) + 2$ is not $K(A)$-valid ($K(A)^C$-valid) if and only if there is an $FK(e_n)$-tree-model $(FK^C(e_n)$-tree-model) $\mathfrak{M}^*$($e_n$) = $(W_n, R, V, \{\Phi(i)\}_{i \in W_n}, \{\Psi(i)\}_{i \in W_n})$ for which $V(\varphi, 0) < 1$ and the finite system $A(e_n) = \langle A(e_n), \bigcup_{i \in W_n} \Phi(i), \bigcup_{i \in W_n} \Psi(i) \rangle$ is consistent with $A$.

Choose non-deterministically $V$: $\text{Var}(\varphi) \to A(e_n)$, $R: W_n^2 \to A(e_n)$, and $\Phi(i), \Psi(i) \subseteq A(e_n)^2$ for all $i \in W_n$ to obtain $\mathfrak{M}^*$($e_n$), and compute $V(\varphi, 0)$ to verify $V(\varphi, 0) < 1$. This takes a number of steps bounded by a constant multiple of the exponential $e_n$. Then utilize an oracle to verify the consistency of $A(e_n)$ with $A$. \hfill \Box

Example 19. Any finite system $A(m)$ is consistent with $G$: $A(m) = \langle A(m), \Phi, \Psi \rangle$ is consistent with $G_i$ if and only if $\Psi = \emptyset$ and $\Phi = \{(0, \frac{k_1}{m}), \ldots, (0, \frac{k_l}{m})\}$ for some $l \in \mathbb{N}^+$ and $k_1, \ldots, k_l \in \mathbb{N}$, or is $\emptyset$, and $A(m)$ is consistent with $G^\uparrow$ if and only if $\Phi = \emptyset$ and $\Psi = \{(\frac{k_1}{m}, 1), \ldots, (\frac{k_l}{m}, 1)\}$ for some $l \in \mathbb{N}^+$ and $k_1, \ldots, k_l \in \mathbb{N}$, or is $\emptyset$. Therefore, in these cases the consistency problem is obviously decidable in linear time and space (null-space if the size of the input tape is not considered).

Moreover, it is easy to verify inductively that any algebra $A$ obtained from $G$, $G_i$, $G^\uparrow$, and finite order-based algebras as a finite combination of ordered sums, lexicographical products, and fusion of consecutive points has a (PTIME) decidable consistency problem. In all of these cases, validity in $K(A)$ and $K(A)^C$ is (co-NEXPTIME) decidable. This includes the case when $A$ as an ordered set is isomorphic to any ordinal $\alpha + 1 < \omega^\omega$ or its reverse.

The algebras $G$, $G_i$, $G^\uparrow$, and the finite order-based algebras have the additional property that if the finite systems $\langle A(m), \Phi_i, \Psi_i \rangle$, for $i = 0, \ldots, k$, are consistent with $A$, then the same holds for $\langle A(m), \bigcup_{i \leq k} \Phi_i, \bigcup_{i \leq k} \Psi_i \rangle$. This will allow us to improve the decidability result in these cases to PSPACE-completeness. However, we first need a result about $FK(e_n)$-tree-models.

Lemma 20. The following problem is PSPACE-reducible (in $n$) to the consistency of finite systems with $A$:

Given $\Sigma = \{\varphi_1, \ldots, \varphi_k\} \subseteq Fm$ (not necessarily distinct formulas) such that $k \leq n$ and $\ell(\varphi_j) \leq n$ for all $1 \leq j \leq k$, and given intervals $I_1, \ldots, I_k \subseteq A(e_n)$ (closed or open at their endpoints), determine if there exists a (crisp) $FK(e_n)$-tree-model $\mathfrak{M}^* = \langle W_n, R, V, \{\Phi(i)\}_{i \in W_n}, \{\Psi(i)\}_{i \in W_n} \rangle$, with root $0$ and height
Since PSPACE = NPSPACE (see [31]), it suffices to give a non-deterministic polynomial space proof. Let $M$ be a polynomial space algorithm to produce a witnessing model for $K$. Let $\Sigma$ be the complexity of the classical modal logic $R_{xy}$ and values at accumulation points. Then $\log n$ is a witness of right homogeneity at $a$. Choose partial functions $\Phi(x) = \{ \langle a, c_a \rangle : a \in G \} \subseteq V[S_{\Box}, x] \times A(e_n)$ and $\Psi(x) = \{ \langle d_b, b \rangle : b \in H \} \subseteq V[S_{\Diamond}, x] \times A(e_n)$ and verify that the finite system $\langle A(e_n), \Phi(x), \Psi(x) \rangle$ is consistent with $A$. Each $a \in G$ plays the role of a “right accumulation point” and $c_a$ plays the role of a “witness of right homogeneity” at $a$, and similarly for $H$. An oracle for the consistency problem must certify that this distribution can be realized in $A$.

Choose also worlds $y_1, \ldots, y_m \in W_n$ for $m \leq n$ in the next level of the tree and values $R_{xyt} \in A(e_n)$ for $t = 1, \ldots, m$.

Proof. Since PSPACE = NPSPACE (see [31]), it suffices to give a non-deterministic polynomial space algorithm to produce a witnessing model for $\mathcal{M}(e_n)$. Since the full model may need exponential space to be displayed, our strategy is to search sequentially the branches of $\mathcal{M}(e_n)$, from the root down, so that all branches are built in the same polynomial space. This is the basic idea of Ladner’s proof in [24] of the PSPACE-completeness of the classical modal logic $K$. We do not try to optimize the space bound but show that $22n^3$ does the job.

Input. Each value in $A(e_n)$ may be represented by a binary word of length at most $\log e_n \leq 2n^2$, and the only information we need from the input, besides $\Sigma$, is the maximum (strictly smaller than 1) of $A(e_n)$ and the endpoints of the intervals $I_j$, indicating if they are included or not in the intervals. We consider also as part of the input, a particular world $x \in W_n$, written in binary notation (length $\leq \log n^2 \leq n^2$). At the initial stage, $x = 0$, with appropriate markings in the formulas, we may assume further that each $\varphi_j$ appears decomposed in the form:

$$\varphi_j = \chi_j(p_1, \ldots, p_l, \Box \psi^j_1, \ldots, \Box \psi^j_{n_j}, \Diamond \theta^j_1, \ldots, \Diamond \theta^j_{m_j}),$$

where $P = \{ p_1, \ldots, p_l \} \subseteq \text{Var}$ and $\chi_j(p_1, \ldots, p_l, q_1, \ldots, q_{n_j}, s_1, \ldots, s_{m_j})$ is a non-modal formula. Set:

$$S_{\Box} = \{ \Box \psi^j_1, \ldots, \Box \psi^j_{n_j} : j = 1, \ldots, k \}, \quad S_{\Diamond} = \{ \Diamond \theta^j_1, \ldots, \Diamond \theta^j_{m_j} : j = 1, \ldots, k \}$$

$$F_{\Box} = \{ \psi^j_1, \ldots, \psi^j_{n_j} : j = 1, \ldots, k \}, \quad F_{\Diamond} = \{ \theta^j_1, \ldots, \theta^j_{m_j} : j = 1, \ldots, k \}.$$

Comment. It should be clear that the input may be displayed in space at most $3n^2 + (1 + 2n)2n^2 \leq 9n^3$.

Step 1. Choose values $V(\rho, x) \in A(e_n)$, for all $\rho \in P \cup S_{\Box} \cup S_{\Diamond}$, and verify that $V(\varphi_j, x) \in I_j$ for each $j \leq k$.

Choose partial functions $\Phi(x) = \{ \langle a, c_a \rangle : a \in G \} \subseteq V[S_{\Box}, x] \times A(e_n)$ and $\Psi(x) = \{ \langle d_b, b \rangle : b \in H \} \subseteq V[S_{\Diamond}, x] \times A(e_n)$ and verify that the finite system $\langle A(e_n), \Phi(x), \Psi(x) \rangle$ is consistent with $A$. Each $a \in G$ plays the role of a “right accumulation point” and $c_a$ plays the role of a “witness of right homogeneity” at $a$, and similarly for $H$. An oracle for the consistency problem must certify that this distribution can be realized in $A$.

Choose also worlds $y_1, \ldots, y_m \in W_n$ for $m \leq n$ in the next level of the tree and values $R_{xyt} \in A(e_n)$ for $t = 1, \ldots, m$.
Comment. The space required to perform the previous step and store the data produced is at most \(3n \cdot 2n^2 + n \cdot n^2 = 7n^3\). This step guesses the values of the desired tree-model \(\mathfrak{M}^*_t\) at the root. Hence, this model exists if and only if it is possible to find further (crisp, if necessary) \(\text{FK}(e_n)\)-tree-models \(\mathfrak{M}^*_t\) of height \(\leq n - 1\) with respective roots \(y_t\), for \(t = 1, \ldots, m\), such that for any \(\rho \in F^\square \cup F^\triangledown\),

1. \(\bigwedge_{t=1}^m (Rx y_t \rightarrow V(\rho, y_t)) \in [V(\square \rho, x), c_a)\) if \(\rho \in F^\square\) and \(V(\square \rho, x) = a \in G\),
2. \(\bigwedge_{t=1}^m (Rx y_t \rightarrow V(\rho, y_t)) = V(\square \rho, x)\) if \(\rho \in F^\square\) and \(V(\square \rho, x) \notin G\),
3. \(\bigvee_{t=1}^m (Rx y_t \wedge V(\rho, y_t)) \in (d_b, V(\triangle \rho, x)]\) if \(\rho \in F^\triangledown\) and \(V(\triangle \rho, x) = b \in H\),
4. \(\bigvee_{t=1}^m (Rx y_t \wedge V(\rho, y_t)) = V(\triangle \rho, x)\) if \(\rho \in F^\triangledown\) and \(V(\triangle \rho, x) \notin H\).

If \(F^\square_t (F^\triangledown_t)\) denotes the set of \(\rho \in F^\square (\rho \in F^\triangledown)\) for which the minimum (maximum) associated to \(\rho\) above is realized at \(y_t\), then the above conditions are equivalent to asking for all \(t\) and \(\rho\):

1. \(Rx y_t \rightarrow V(\rho, y_t) \geq V(\square \rho, x)\) if \(\rho \in F^\square\)
2. \(Rx y_t \rightarrow V(\rho, y_t) \in [V(\square \rho, x), \delta_\rho)\) if \(\rho \in F^\square_t\) and \(V(\square \rho, x) \in G\),
3. \(Rx y_t \rightarrow V(\rho, y_t) = V(\square \rho, x)\) if \(\rho \in F^\square_t\) and \(V(\square \rho, x) \notin G\),
4. \(Rx y_t \wedge V(\rho, y_t) \leq V(\triangle \rho, x)\) if \(\rho \in F^\triangledown\)
5. \(Rx y_t \wedge V(\rho, y_t) \in (\mu_\rho, V(\triangle \rho, x)]\) if \(\rho \in F^\triangledown_t\) and \(V(\triangle \rho, x) \in H\),
6. \(Rx y_t \wedge V(\rho, y_t) = V(\triangle \rho, x)\) if \(\rho \in F^\triangledown_t\) and \(V(\triangle \rho, x) \notin H\).

These conditions are equivalent, in turn, to asking for each model \(\mathfrak{M}^*_t\) and \(\rho \in F^\square \cup F^\triangledown\) the value \(V(\rho, y_t)\) to belong to the interval \(I_{\rho,t}\), determined as
for the algebras $G$

Theorem 21. The validity problems for $K(A)$ and $K(A)^c$ are PSPACE-complete for the algebras $G$, $G_\downarrow$, and $G_\uparrow$. 

1. $[V(\Box \rho, x), Rxy_t) \quad if \rho \in F_\Box \setminus F_\Box^t$ and $V(\Box \rho, x) < 1,$
   $[Rxy_t, 1] \quad if \rho \in F_\Box \setminus F_\Box^t$ and $V(\Box \rho, x) = 1,$

2. $[V(\Box \rho, x), c_a \land Rxy_t) \quad if \rho \in F_\Box^t$ and $V(\Box \rho, x) = a \in G,$

3. $[V(\Box \rho, x), V(\Box \rho, x)] \quad if \rho \in F_\Box^t, V(\Box \rho, x) \notin G$ and $V(\Box \rho, x) < 1,$
   $[Rxy_t, 1] \quad if \rho \in F_\Box^t, V(\Box \rho, x) \notin G$ and $V(\Box \rho, x) = 1,$

4. $[0, V(\Diamond \rho, x)] \quad if \rho \in F_\Diamond \setminus F_\Diamond^t$ and $Rxy_t > V(\Diamond \rho, x),$

5. $(d_b, V(\Diamond \rho, x)] \quad if \rho \in F_\Diamond^t$ and $V(\Diamond \rho, x) = b \in H,$

6. $[V(\Diamond \rho, x), V(\Diamond \rho, x)] \quad if \rho \in F_\Diamond^t$ and $V(\Diamond \rho, x) \notin H.$

(Why can we ignore the value $Rxy_t$ in the cases 5 and 6? And what if $Rxy_t \leq V(\Diamond \rho, x)$ in case 4? )

But this amounts to the original problem: the existence of $M'_t$ with root $y_t$

satisfying the conditions of the Lemma for the input $\Sigma' = F_\Box \cup F_\Diamond$ and intervals $I_{\rho,t}, \rho \in \Sigma'.$ This justifies the next steps of the algorithm.

Step 2. Find coverings $F_\Box = \bigcup_{t \in [1,m]} F_\Box^t$ and $F_\Diamond = \bigcup_{t \in [1,m]} F_\Diamond^t$ and compute

for each $t$ and $\rho \in F_\Box \cup F_\Diamond$ the interval $I_{\rho,t}.$

Comment. Computing and storing the data produced in this step requires space at most $2n \cdot n^2 + 2n^2 \cdot 2n^2 = 6n^4.$

Step 3. For $t = 1, \ldots, m$, return consecutively to Step 1 with input: $\Sigma' = F_\Box \cup F_\Diamond, \{I_{\rho,t} : \rho \in \Sigma'\},$ and $x = y_t,$ traversing the resulting tree of worlds in pre-order; that is, the leftmost branch is exhausted before passing to the next unexplored sub-branch at the right.

Comment. The cyclic repetition of Steps 1 and 2 (an exponential number of times), if successful at each stage, runs through a tree of height equal to the maximum modal depth $D$ of the original $\varphi_j.$ Since each cycle takes space at most $9n^3 + 7n^3 + 6n^4 \leq 22n^4,$ the space needed to guess a branch of the tree is at most $22n^4 \cdot D \leq 22n^5.$ The key point is that having verified successfully the existence of a branch we may utilize the same space for the next one, and thus the total space required is bounded by $22n^5.$ Informally, returning to Step 1 with $t = 1$ starts a search for $M'_1,$ after finishing it successfully, one goes to Step 1 with $t = 2$ and utilize the same space, bounded by $22n^4(n - 1)$, to search for $M'_2,$ etc. Adding to this common space the space of the first cycle we obtain $22n^5.$

\[\square\]

Theorem 21. The validity problems for $K(A)$ and $K(A)^c$ are PSPACE-complete for the algebras $G$, $G_\downarrow$, and $G_\uparrow.$
Proof. Lemma 20 applied to a formula $\varphi$ and the interval $I = [0, 1)$ yields a PSPACE algorithm on the length of $\varphi$ to determine for these algebras, whether there is a $\text{FK}(e_n)$-tree-model for which $V(\varphi, 0) < 1$ and $\langle A(e_n), \Phi(i), \Psi(i) \rangle$ is consistent with $A$, for each $i \in W_n \subseteq \{0, 1, \ldots, n^n\}$. Since the latter condition is equivalent, for these algebras, to consistency with $A$ of $\langle A(e_n), \bigcup_i \Phi(i), \bigcup_i \Psi(i) \rangle$. The existence of this model is equivalent to the existence of a $K(A)$-counter-model for $\varphi$, by the initial discussion of the section. The lower bound follows from the fact that classical modal logic $K$ is PSPACE-hard [24] and can be interpreted faithfully in $K(A)$ or $K(A)^C$ by the double negation interpretation which adds $\neg\neg$ in front of any subformula of a formula.

Remark. The last theorem applies to any algebra for which the consistency problem is PSPACE decidable and the union of consistent finite systems is consistent. Examples of these algebras are finite algebras (trivially), the ordinals $\omega^n + 1$, $n \in \mathbb{N}^+$, and their reverse orders. We also expect that PSPACE-completeness holds for all finite combinations of $G$, $G^\downarrow$, $G^\uparrow$, and finite algebras built via ordered sums, lexicographical products, and fusion of consecutive points, but will not prove this here.

To generalize the results in this section to languages with a finite set of constants $C_L = \{c_1 < \ldots < c_l\}$, utilize a set of values $A'(e_n)$ containing an isomorphic copy $C'_L = \{c'_1 < \ldots < c'_l\}$ of $C_L$ such that $\lvert [c_i', c_{i+1}]A'(e_n) \rvert = \lvert [c_i, c_{i+1}]A \rvert$, if $\lvert [c_i, c_{i+1}]A \rvert < e_n$, and $\lvert [c_i', c_{i+1}]A'(e_n) \rvert = e_n$, otherwise. This allows $V$ and $R$ to take values in any possible interval of consecutive constants. Moreover, $\lvert A'(e_n) \rvert \leq \lvert C_L \rvert e_n$ and all bounds are multiplied by a constant. Finite systems must have now the form $\langle A(m), \Phi, \Psi, \{c'\}_{c \in C_L} \rangle$ and the embeddings granting consistency must send $c'$ to $c$.

6. Order-Based Crisp S5 Logics

As in the classical setting, further many-valued modal logics may be defined for a given order-based algebra $A$ as logics of particular classes of $K(A)$-models (see, e.g., [9, 10]). In this section, we restrict our attention to proving decidability and co-NP-completeness for crisp “S5” order-based logics that may be understood also as one-variable fragments of order-based first-order logics. In particular, we give a positive answer to the open decidability problem (and establish co-NP-completeness) for validity in the one-variable fragment of first-order Gödel logic (see, e.g., [18, Chapter 9, Problem 13]).
We define an $S_5(A)^C$-model to be a $K(A)^C$-model $\mathfrak{M} = \langle W, V, R \rangle$ such that $R$ is an equivalence relation. We call $\mathfrak{M}$ universal if $R = W \times W$ and in this case just write $\mathfrak{M} = \langle W, V \rangle$, noting that the clauses for $\Box$ and $\Diamond$ simplify to

$$V(\Box \varphi, x) = \bigwedge \{ V(\varphi, y) : y \in W \}$$
$$V(\Diamond \varphi, x) = \bigvee \{ V(\varphi, y) : y \in W \}.$$  

The following lemma is an immediate corollary of Lemma 2 and the fact that the submodel of an $S_5(A)^C$-model generated by one world is universal.

**Lemma 22.** $\models_{S_5(A)^C} \varphi$ if and only if $\varphi$ is valid in all universal $S_5(A)^C$-models.

It follows that each order-based modal logic $S_5(A)^C$ may be viewed as the one-variable fragment of a corresponding order-based first-order logic. Rather than define this first-order logic and then restrict to its one-variable fragment, let us simply note that the first-order translation of $\varphi \in \text{Fm}$ is obtained by replacing each propositional variable $p$ with the predicate $p(x)$, $\Box$ with $\forall x$, and $\Diamond$ with $\exists x$. In particular, $S_5(G)^C$ is the Gödel modal logic $GS_5^C$ corresponding to the one-variable fragment of first-order Gödel logic (see, e.g., [1, 18]). $GS_5^C$ is axiomatized in [10] as an extension of the intuitionistic modal logic MIPC studied in [7, 30] with the prelinearity axiom schema and $\Box(\Box \varphi \lor \psi) \rightarrow (\Box \varphi \lor \Box \psi)$. Let us also remark in passing that the logic $GS_5^C$ based on non-crisp frames may be axiomatized as MIPC plus prelinearity [10], and that decidability of the validity problem follows from the finite model property for the semantics with two accessibility relations [3].

The infinite $K(A)$-model defined in the proof of Theorem 7 for the formula $\Box \neg \neg p \rightarrow \neg \neg \Box p$ is a universal $S_5(A)^C$-model. Hence, if the universe of $A$ is $[0, 1]$ or $G_1$ and the language includes $\rightarrow$, then $S_5(A)^C$ does not have the finite model property. Also, as in Theorem 6, the logic $S_5(G_1)^C$ has the finite model property, but not if $\Delta$ is added to the language. We will prove decidability for these and other cases using again a new equivalent semantics.

Let us assume once more that $A$ is a locally homogeneous order-based algebra. We define an $FS_5(A)^C$-model as an $FK(A)^C$-model $\mathfrak{M} = \langle W, R, V, T_\Box, T_\Diamond \rangle$ such that $\langle W, R, V \rangle$ is an $S_5(A)^C$-model, and for all $x, y \in W$:

1. $T_\Box(x) = T_\Box(y)$ and $T_\Diamond(x) = T_\Diamond(y)$ whenever $Rxy$,
2. $\{ V(\Diamond p, x) : p \in \text{Var} \} \subseteq T_\Box(x)$ and $\{ V(\Box p, x) : p \in \text{Var} \} \subseteq T_\Diamond(x)$.
We call $\mathcal{M}$ universal if $R = W \times W$ and in this case write $\mathcal{M} = \langle W, V, T, T_0 \rangle$, where $T$ and $T_0$ may now be understood as fixed subsets of $A$, and the clauses for $\Box$ and $\Diamond$ simplify to

\[
V(\Box \varphi, x) = \bigvee \{ r \in T : r \leq \bigwedge \{ V(\varphi, y) : y \in W \} \}
\]

\[
V(\Diamond \varphi, x) = \bigwedge \{ r \in T_0 : r \geq \bigvee \{ V(\varphi, y) : y \in W \} \}.
\]

Note in particular that in universal $S5(A)^C$-models and $FS5(A)^C$-models, the truth values of box-formulas and diamond-formulas are independent of the world.

The new condition (ii) for $FS5(A)^C$-models reflects the fact that we deal here with universal models not tree models and must therefore take into account the values of diamond-formulas and box-formulas when fixing the values in $T$ and $T_0$, respectively. It is easily shown that (ii) extends inductively for universal $FS5(A)^C$-models to the following condition on all diamond and box formulas:

**Lemma 23.** For a universal $FS5(A)^C$-model $\mathcal{M} = \langle W, V, T, T_0 \rangle$:

\[
\{ \hat{V}(\Diamond \varphi, x) : \varphi \in \text{Fm}, x \in W \} \subseteq T_0 \text{ and } \{ \hat{V}(\Box \varphi, x) : \varphi \in \text{Fm}, x \in W \} \subseteq T.
\]

We now show that $S5(A)^C$-validity is equivalent to validity in finite universal $FS5(A)^C$-models, following fairly closely the corresponding proofs from previous sections.

**Lemma 24.** Let $\Sigma \subseteq \text{Fm}$ be a finite fragment, $\mathcal{M} = \langle W, V \rangle$ a universal $S5(A)^C$-model, and $x \in W$. Then there is a finite universal $FS5(A)^C$-model $\hat{\mathcal{M}} = \langle \hat{W}, \hat{V}, \hat{T}, \hat{T}_0 \rangle$ with $x \in \hat{W} \subseteq W$ and $|\hat{W}| \leq |\Sigma|$ such that $\hat{V}(\varphi, y) = V(\varphi, y)$ for all $\varphi \in \Sigma$ and $y \in \hat{W}$.

**Proof.** The proof is similar to the proof of Lemma 12. Let us fix a finite fragment $\Sigma \subseteq \text{Fm}$, a universal $S5(A)^C$-model $\mathcal{M} = \langle W, V \rangle$, and $x \in W$. Consider the finite (possibly empty) sets

\[
V[\Sigma, x] \cap R(A) = \{ a_i : i \in I \} \quad \text{and} \quad V[\Sigma, x] \cap L(A) = \{ b_j : j \in J \},
\]

noting that these sets are independent of the choice of the world $x \in W$. For each $i \in I$, choose a witness of right homogeneity $c_i$ at $a_i$ such that the intervals $(a_i, c_i)$ are pairwise disjoint for all $i \in I$, and

\[
(V[\Sigma, x] \cup \{ V(\Diamond p, x) : p \in \text{Var} \} \cup C_L) \cap \bigcup_{i \in I} (a_i, c_i) = \emptyset.
\]
Similarly, for each $j \in J$, choose a witness of left homogeneity $d_j$ at $b_j$ such that the intervals $(d_j, b_j)$ are pairwise disjoint for all $j \in J$, and

$$(V[\Sigma_0, x] \cup \{V(\Box p, x) : p \in \text{Var} \cap \Sigma\} \cup C_{\xi}) \cap (\bigcup_{j \in J} (d_j, b_j)) = \emptyset.$$ 

We define

$$\hat{T}_{\Box} = A \setminus \bigcup_{i \in I} (a_i, c_i) \quad \text{and} \quad \hat{T}_{\Diamond} = A \setminus \bigcup_{j \in J} (d_j, b_j).$$

Now consider $\varphi = \Box \psi \in \Sigma_{\Box}$ and $a = V(\Box \psi, x) \in \hat{T}_{\Box}$. If $a \notin R(A)$, then we choose $y_\varphi \in W$ such that $a = V(\psi, y_\varphi)$. If $a \in R(A)$, then there is an $i \in I$ such that $a = a_i$, and we choose $y_\varphi \in W$ such that $V(\psi, y_\varphi) \in [a_i, c_i]$. Suppose now that $\varphi = \Diamond \psi \in \Sigma_{\diamond}$ and $b = V(\Diamond \psi, x) \in \hat{T}_{\Diamond}$. If $b \notin L(A)$, then we choose $y_\varphi \in W$ such that $b = V(\psi, y_\varphi)$. If $b \in L(A)$, then there is a $j \in J$ such that $b = b_j$, and we choose $y_\varphi \in W$ such that $V(\psi, y_\varphi) \in (d_j, b_j]$.

Now let $\widehat{W} = \{x\} \cup \{y_\varphi \in W : \varphi \in \Sigma_{\Box} \cup \Sigma_{\diamond}\}$, noting that $|\widehat{W}| \leq 1 + |\Sigma_{\Box} \cup \Sigma_{\diamond}| \leq |\Sigma|$. Define for each $y \in \widehat{W}$ and $p \in \text{Var}$:

$$\widehat{V}(p, y) = \begin{cases} V(p, y) & \text{if } p \in \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\hat{M} = \langle \widehat{W}, \widehat{V}, \hat{T}_{\Box}, \hat{T}_{\Diamond} \rangle$ is a finite $\text{FS5}(A)^C$-model satisfying $x \in \widehat{W} \subseteq W$ and $|\widehat{W}| \leq |\Sigma|$. It then follows by an easy induction on $\ell(\varphi)$ that $\widehat{V}(\varphi, y) = V(\varphi, y)$ for all $y \in \widehat{W}$ and $\varphi \in \Sigma$.

Note that the number of intervals omitted from $\hat{T}_{\Box}$ and $\hat{T}_{\Diamond}$, defined in Lemma 24, is smaller than or equal to the cardinality of $\Sigma_{\Box}$ and $\Sigma_{\diamond}$, respectively, for the given fragment $\Sigma$.

**Lemma 25.** Let $\mathcal{M} = \langle W, V, T_{\Box}, T_{\Diamond} \rangle$ be a finite universal $\text{FS5}(A)^C$-model. Then there is a universal $\text{S5}(A)^C$-model $\mathcal{W} = \langle \widehat{W}, \widehat{V} \rangle$ with $W \subseteq \widehat{W}$ such that $\widehat{V}(\varphi, x) = V(\varphi, x)$ for all $\varphi \in \text{Fm}$ and $x \in W$.

**Proof.** Given a finite universal $\text{FS5}(A)^C$-model $\mathcal{M}$, we construct our universal $\text{S5}(A)^C$-model $\hat{M}$ directly by taking infinitely many copies of $\mathcal{M}$.

Consider $T_{\Box} = A \setminus \bigcup_{i \in I} (a_i, c_i)$ and $T_{\Diamond} = A \setminus \bigcup_{j \in J} (d_j, b_j)$ for finite (possibly empty) sets $I, J$, where for each $i \in I$, right homogeneity at $a_i \in R(A)$ is witnessed by $c_i$ such that the intervals $(a_i, c_i)$ are pairwise disjoint, and, similarly, for
each \( j \in J \), left homogeneity at \( b_j \in L(A) \) is witnessed by \( d_j \) such that the intervals \((d_j, b_j)\) are pairwise disjoint. We define a family of \( C_L \)-order embeddings \( \{h_k: A \to A\}_{k \in \mathbb{N}^+} \) such that

- for each even \( k \in \mathbb{N}^+ \), \( h_k \) is the identity function on \( T_\square \) and for each \( i \in I \),
  \[ h_k[[a_i, c_i]] \subseteq [a_i, a_i + \frac{1}{k}] \),

- for each odd \( k \in \mathbb{N} \), \( h_k \) is the identity function on \( T_\Diamond \) and for each \( j \in J \),
  \[ h_k[(d_j, b_j)] \subseteq (b_j - \frac{1}{k}, b_j) \).

Note that Lemma 23 ensures for all \( x \in W \) that \( V[\Sigma_\square \cup \Sigma_\Diamond, x] \subseteq T_\square \cap T_\Diamond \) and hence that for all \( k \in \mathbb{N}^+ \) (even and odd), \( h_k \) is the identity function on \( V[\Sigma_\square \cup \Sigma_\Diamond, x] \). Let \( h_0 \) be the identity on \( A \), let \( \hat{W}_0 = W \), and for each \( k \in \mathbb{N}^+ \), let \( \hat{W}_k \) be a copy of \( W \) with a distinct copy \( \hat{x}_k \) of each \( x \in W \); also let \( \hat{x}_0 = x \) for each \( x \in W \). We define the universal \( S5(A)^C \)-model \( \hat{M} = (\hat{W}, \hat{V}) \) where

\[ \hat{W} = \bigcup_{k \in \mathbb{N}} \hat{W}_k \quad \text{and} \quad \hat{V}(p, \hat{x}_k) = h_k(V(p, x)) \quad \text{for all} \quad p \in \text{Var}, x \in W, \quad \text{and} \quad k \in \mathbb{N}. \]

It suffices now to prove that for all \( \varphi \in Fm, x \in W, \quad \text{and} \quad k \in \mathbb{N}, \)

\[ \hat{V} (\varphi, \hat{x}_k) = h_k(V(\varphi, x)) \]

proceeding by induction on \( \ell(\varphi) \). The base case follows by definition, while for the non-modal connectives, the argument is the same as in the proof of Lemma 1. Consider \( \varphi = \Diamond \psi \). Fix \( x \in W \) and \( k \in \mathbb{N} \). There are two cases.

(a) Suppose that \( V(\Diamond \psi, x) = b_j \) for some \( j \in J \). Note first that by Lemma 23, \( V(\Diamond \psi, x) = b_j \in T_\square \) and hence \( h_k(b_j) = b_j \). Clearly \( V(\psi, z) \leq b_j \) for all \( z \in W \). Hence, by the induction hypothesis and the construction of \( \{h_n: A \to A\}_{n \in \mathbb{N}} \), for all \( n \in \mathbb{N} \) and \( \hat{z}_n \in \hat{W}, \)

\[ \hat{V}(\psi, \hat{z}_n) = h_n(V(\psi, z)) \leq b_j. \]

Also, for some \( y \in W, \)

\[ V(\psi, y) \in (d_j, b_j). \]

Hence for any odd \( n \in \mathbb{N}, \)

\[ h_n(V(\psi, y)) \in (b_j - \frac{1}{n}, b_j). \]
Using the induction hypothesis,

\[ \tilde{V}(\diamond \psi, \bar{x}_k) = \bigvee \{ \tilde{V}(\psi, \bar{y}_n) : y \in W, n \in \mathbb{N} \} \]
\[ = \bigvee \{ h_n(V(\psi, y)) : y \in W, n \in \mathbb{N} \} \]
\[ = \bigvee \{ b_j - \frac{1}{n} : n \in \mathbb{N}^+ \} \]
\[ = \ b_j. \]

So \( \tilde{V}(\diamond \psi, \bar{x}_k) = b_j = h_k(b_j) \) as required.

(b) Suppose that \( V(\diamond \psi, x) = b \neq b_j \) for all \( j \in J \). Note again that by Lemma 23, \( V(\diamond \psi, x) = b \in T_\Box \) and hence \( h_k(b) = b \). Clearly, \( V(\psi, z) \leq b \) for all \( z \in W \). It follows again by the induction hypothesis and the construction of \( \{ h_n : A \to A \}_{n \in \mathbb{N}} \) that for all \( n \in \mathbb{N} \) and \( \bar{z}_n \in \tilde{W} \),

\[ \tilde{V}(\psi, \bar{z}_n) = h_n(V(\psi, z)) \leq b. \]

Moreover, because \( W \) is finite, there is a \( y \in W \) such that \( V(\psi, y) = b = V(\diamond \psi, x) \). Using the induction hypothesis and the fact that \( h_n \) is the identity function on \( V[\Sigma_\Box \cup \Sigma_0, z] \) for all \( n \in \mathbb{N} \) and all \( z \in W \), it follows that

\[ \tilde{V}(\diamond \psi, \bar{x}) = \bigvee \{ \tilde{V}(\psi, \bar{z}_n) : z \in W, n \in \mathbb{N} \} \]
\[ = \bigvee \{ h_n(V(\psi, z)) : z \in W, n \in \mathbb{N} \} \]
\[ = \bigvee \{ h_n(b) : n \in \mathbb{N} \} \]
\[ = b \]
\[ = h_k(V(\diamond \psi, x)). \]

The case \( \varphi = \Box \psi \) is very similar. \( \square \)

Combining Lemmas 22, 24, and 25, we obtain the following equivalence.

**Theorem 26.** Let \( A \) be a locally homogeneous order-based algebra. Then \( \models_{\text{SS}(A)^c} \varphi \) if and only if \( \varphi \) is valid in all finite universal FS5(A)^c-models.

The desired decidability and complexity results are now obtained by considering the number of truth values needed to check validity of formulas in finite universal FS5(A)^c-models. Recall (see Section 5) that if \( A(m) = \{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1\} \), then a finite system \( \langle A(m), \Phi, \Psi, \{c'_e \in C_e \rangle \), where \( \Phi, \Psi \subseteq A(m)^2 \), is consistent.
with \( \mathbf{A} \) if there exists an order preserving embedding \( h: A(m) \to A \) such that 
\[
h(c') = c \quad \text{for all } c \in C_L, \quad h(c) \quad \text{witnesses right homogeneity at } \quad h(a) \in R(\mathbf{A}), \]
for all \( \langle a, c \rangle \in \Phi \), and \( h(d) \) witnesses left homogeneity at \( h(b) \in L(\mathbf{A}) \), for all \( \langle d, b \rangle \in \Phi \).

**Theorem 27.** Let \( \mathbf{A} \) be a locally homogeneous order-based algebra. Then the validity problem of \( S5(\mathbf{A})^C \) is co-NP reducible to the problem of consistency of finite systems with \( \mathbf{A} \).

**Proof.** Consider \( \varphi \in F_m \) and let \( n = \ell(\varphi) + |C_L| \). To check if \( \varphi \) is not \( S5(\mathbf{A})^C \)-valid, it suffices, by Lemmas 24 and 25, to check if \( \varphi \) is not valid in a finite universal \( FS5(\mathbf{A})^C \)-model \( \mathcal{M} = \langle W, V, T^\square, T^\diamond \rangle \) with \( |W| \leq |\Sigma(\varphi)| \leq \ell(\varphi) + |C_L| = n \). To compute \( V(\varphi, x) \) in such a model, we need to know only the values \( V[\Sigma(\varphi), W] \) (that is, fewer than \( n^2 \) values) and the endpoints of the intervals defining \( T^\square \) and \( T^\diamond \) (that is, fewer than \( 2n \) values). So, we need at most \( 3n^2 \) distinct values. Therefore, we may assume that these values are in a fixed finite set \( A_n = A(p(n)) = \{0, \frac{1}{p(n)}, \ldots, \frac{p(n) - 1}{p(n)}, 1\} \), containing properly spaced copies of constants, where \( p(n) = 3|C_L|n^2 \). We may assume also that \( W = W_n \subseteq \{0, 1, \ldots, n - 1\} \). Then checking non-deterministically that \( \varphi \) is not valid amounts to performing the following steps:

1. Guessing the values \( V(p, i) \) in \( A_n \) for each \( p \in \text{Var}(\varphi) \) and \( i \in W_n \) (at most \( np(n) \) steps).
2. Guessing the sets \( \Phi, \Psi \subseteq A_n^2 \) such that \( \Phi \) and \( \Psi \) define families of disjoint open intervals and using them to define, respectively, \( T^\square, T^\diamond \subseteq A(p(n)) \) (at most \( 2p(n)^2 \) steps).
3. Checking that the system \( \langle A_n, \Phi, \Psi, \{a_c\}_{c \in C_L} \rangle \) is consistent with \( \mathbf{A} \).
4. Computing \( V(\varphi, 0) \) in the model \( \langle W_n, V, T^\square, T^\diamond \rangle \) and checking that \( V(\varphi, 0) < 1 \) (essentially \( n^3 \) steps).

Therefore, a counter-model for \( \varphi \) may be guessed in polynomial time if we have an oracle for the consistency problem. \( \square \)

**Corollary 28.** The validity problems of \( S5(\mathbf{G})^C, S5(\mathbf{G}_\downarrow)^C, \) and \( S5(\mathbf{G}_\uparrow)^C \) are co-NP-complete. The same is true for \( S5(\mathbf{A})^C \) if \( \mathbf{A} \) is a finite combination of \( \mathbf{G}, \mathbf{G}_\downarrow, \) \( \mathbf{G}_\uparrow, \) and finite algebras via ordered sums, lexicographical products, and fusion of consecutive points.

**Proof.** The validity problem is co-NP hard already for the pure propositional logic over any \( \mathbf{A} \), because classical propositional logic is interpretable in these logics.
Moreover, for $G$, $G_{↓}$, and $G_{↑}$, the consistency problem is checked in null or linear time. In the other cases, the consistency problem is solvable in polynomial time.

For crisp Gödel S5 logics and the corresponding one-variable fragments of first-order Gödel logics, the validity problem is co-NP-hard (see [18]) and hence we may conclude the following.

**Theorem 29.** The validity problems of the one-variable fragment of first-order Gödel logics based on $G$, $G_{↓}$, and $G_{↑}$ are co-NP complete. The same is true for the one-variable fragments of first-order Gödel logics based on a finite combination of $G$, $G_{↓}$, and $G_{↑}$, and finite algebras via ordered sums, lexicographical products, and fusion of consecutive points.

7. Concluding Remarks

In this paper, we have established the decidability and PSPACE-completeness of the validity problem for certain “order-based” modal logics, including the Gödel modal logics investigated in [6, 9, 10, 26]. We have also established decidability and co-NP-completeness for the validity problem of “crisp S5” versions of these logics corresponding to one-variable fragments of first-order logics. In particular, we have answered positively the open problem of the decidability (indeed, co-NP-completeness) of the validity problem for the one-variable fragment of first-order Gödel logic. There remain, however, a number of significant questions, notably:

- **Are order-based multi-modal logics also decidable?** This question is of particular interest as many-valued description logics (see, e.g., [5, 20, 34]) may be viewed as many-valued multi-modal logics. The challenge in this case is to extend the new semantics to a multi-modal setting.

- **Is the new semantics suitable for other classes of order-based modal logics?** We have focussed in this paper on “K” and “S5” order-based modal logics, but it would be useful to develop a more general approach that encompasses also decidability for logics based on frames satisfying combinations of conditions such as reflexivity, symmetry, and transitivity.

- **Is validity in the two-variable fragment of first-order Gödel logic decidable?** Notably, validity in the two-variable fragment of first-order classical logic (indeed, any first-order tabular intermediate logic) is decidable [28], while the same fragment of first-order intuitionistic logic is undecidable [23].
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