The Use of the Tukey’s $g - h$ family of distributions to Calculate Value at Risk and Conditional Value at Risk

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Introduction

When calculating the Value at Risk (VaR), there is a little importance to the most extreme losses, since which is not adequately reflect the skewness and kurtosis of the distribution. Moreover, normality is assumption to overestimate the VaR values for upper percentiles, while it underestimates for lower percentiles of values which correspond to more extreme events. In this article we propose use of the Tukey's $g-h$ family of distributions relevant for the calculation of VaR and Conditional Value at Risk (CVaR), as this distribution enables the skewness and kurtosis. We have also calculated explicit formula for CVaR using a Cornish-Fisher expansion to approximate the percentiles. An illustrative example is presented to compare with others models.
Introduction

When calculating the Value at Risk (VaR) there is a little importance to the most extreme losses, since which is not adequately reflect the skewness and kurtosis of the distribution. Moreover, normality is assumption to overestimate the VaR values for upper percentiles, while it underestimates for lower percentiles of values which correspond to more extreme events. In this article we propose use of the Tukey’s $g - h$ family of distributions relevant for the calculation of VaR and Conditional Value at Risk CVaR, as this distribution enables the skewness and kurtosis. We have also calculated explicit formula for CVaR using a Cornish-Fisher expansion to approximate the percentiles. An illustrative example is presented to compare with others models.
Let $Z$ be a random variable with standard normal distribution and $g$ and $h$ are two constants (parameters). The random variable $Y$ given by

$$Y = T_{g, h}(Z) \neq 0, h \in \mathbb{R},$$

(2.1)

has Tukey's $g$–$h$ distribution.

Observations

1. When $h = 0$ the Tukey's $g$–$h$ distribution reduces to $T_{g, 0}(Z) = 1$ $g(exp(gZ) - 1)$ (2.2) which is Tukey's $g$ distribution.

2. Similarly, when $g \to 0$ the Tukey's $g$–$h$ distribution is given by $T_{0, h}(Z) = Z exp\{hZ^2/2\}$ (2.3) known as the Tukey's $h$ distribution.
Tukey’s $g − h$ family of distributions

Let $Z$ be a random variable with standard normal distribution and $g$ and $h$ are two constants (parameters). The random variable $Y$ given by

$$Y = T_{g,h}(Z) = \frac{1}{g} (\exp(gZ) - 1) \exp(hZ^2/2)$$

with $g \neq 0$, $h \in \mathbb{R}$, \hspace{1cm} (2.1)

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2. Similarly, when $g \to 0$ the Tukey’s $g - h$ distribution is given by

$$T_{0,h}(Z) = Z \exp\{hZ^2/2\}$$

(2.3)
Ordinary moments

Martínez and Iglewicz (1984) establishing the $n$–th moments of Tukey’s $g − h$ family distributions, when $h < \frac{1}{n}$, as follows

$$
\mu'_n = \mathbb{E}(Y^n) = \begin{cases} 
\frac{1}{g^n \sqrt{1 - nh}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \exp \left\{ \frac{1}{2} \left( \frac{n-k}{\sqrt{1-nh}} g \right)^2 \right\} & g \neq 0 \\
0 & \text{for } n \text{ odd} \\
\frac{n!}{2^{n/2} (n/2)! \sqrt{(1-nh)^{n+1}}} & \text{for } n \text{ even}, \quad g = 0 
\end{cases}
$$

(2.4)

then the mean of the Tukey's $g − h$ distribution given by

$$
\mu_{g,h} = \mathbb{E}[T_{g,h}(Z)] = \begin{cases} 
\frac{1}{g \sqrt{1-h}} \left[ \exp \left\{ \frac{1}{2} \frac{g^2}{1-h} \right\} - 1 \right] & g \neq 0, \ 0 \leq h < 1, \\
0 & g = 0.
\end{cases}
$$

(2.5)
Table 1 shows the values of \( g \) and \( h \) that approximate a selected set of well known distributions.

<table>
<thead>
<tr>
<th>Name of Distribution</th>
<th>Parameters</th>
<th>Estimates Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( A )</td>
</tr>
<tr>
<td>Cauchy</td>
<td>( \mu, \sigma &gt; 0 )</td>
<td>( \mu )</td>
</tr>
<tr>
<td>Exponential</td>
<td>( \lambda &gt; 0 )</td>
<td>( \frac{1}{\lambda} \ln 2 )</td>
</tr>
<tr>
<td>Laplace</td>
<td>( \alpha, \beta &gt; 0 )</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>Logistic</td>
<td>( \alpha, \beta &gt; 0 )</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>Log-normal</td>
<td>( \mu, \sigma^2, C &gt; 0 )</td>
<td>( C^\mu )</td>
</tr>
<tr>
<td>Normal</td>
<td>( \mu, \sigma^2 )</td>
<td>( \mu )</td>
</tr>
<tr>
<td>( t_{10} )</td>
<td>( \nu = 10 )</td>
<td>0</td>
</tr>
</tbody>
</table>

*Cuadro: Values of \( g \) and \( h \) for some distributions*
Method of quantiles

We start by the method proposed in Hoaglin (1985) which takes into account the properties presented in Dutta and Babbel (2002) and is complemented by the proposal given in Jiménez (2004).

1. \( X \) is a strictly increasing transformation of \( Z \). This is the transformation of a normal standard of \( g - h \) is one to one.

2. The location parameter of the Tukey’s \( g - h \) distribution is estimated by the median of the data, ie \( A = x_{0.5} \).

3. The estimate of the parameter that controls the skewness of the distribution \( (g) \), is estimated, usually by the median of the logarithms of the following expression:

\[
e^{gZ_p} = \frac{UHS_p}{LHS_p}, \quad \forall \ p > 0.5
\]

where \( UHS_p = x_p - x_{0.5} \) and \( LHS_p = x_{0.5} - x_{1-p} \), denote the \( p \)-th upper half-spread and lower half-spread, respectively, defined in Hoaglin et al. (1985).
Method of quantiles

(4) If there is $\theta \in \mathbb{R}$, with $\theta \neq x_{0.5}$, such that

$$\frac{x_p - \theta}{\theta - x_{0.5}} = \frac{x_{0.5} - \theta}{\theta - x_{1-p}} \quad p > 0.5,$$

then $h = 0$. In particular, the expression (3.2) is satisfied if $\theta = A - \frac{B}{g}$. This constant is known as a “threshold parameter” and was obtained in Hoaglin (1985).

(5) When $g \neq 0$, the parameter that controls the elongation (or kurtosis) of the tails ($h$), can be estimated conditionally on this value of $g$

$$\ln (x_{0.5} - \theta_p) = \ln \left(\frac{B}{g}\right) + h\frac{Z_p^2}{2},$$

(3.3)

where $\theta_p < x_{0.5}$ for all $p > 0.5$, and

$$\theta_p = \frac{x_p x_{1-p} - x_{0.5}^2}{UHS_p - LHS_p} \quad \text{for all } p \in (0, 1), p \neq 0.5,$$

(3.4)
Method of moments

The idea of method of moments proposed in Majumder and Ali (2008), to estimate the parameters is to get as many equations as the number of parameters. However, the location and scale parameters are not determined, for this we estimate the parameters $A$ and $B$ as follows

$$A = \mu_X - B \mu_{g,h},$$

(3.6)

where $\mu_X$ is the mean of random variable $X$ and $\mu_{g,h}$ is given in (2.5) and the scale parameter is estimated by

$$B = \text{sgn}(\beta_1(X)) \frac{\sigma_X}{\sigma_Y}.$$

Here $\text{sgn}(\cdot)$ denote the signum function and $\beta_1(X)$ denote the coefficient of skewness from the variable we want to approximate.
Value at Risk (VaR)

Considering a confidence level $\alpha$ and a time horizon of $T$ days, the VaR can be calculated as follows:

$$1 - \alpha = \int_{-\infty}^{\infty} H(S - \text{VaR}_\alpha) g(S) dS$$

where $g(S)$ is the density function of $S$ and $H(u)$ is the Heaviside step function.
### Value at Risk (VaR)

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### Conditional Value at Risk (CVaR)

Conditional Value at Risk CVaR is defined as the expected loss given that is larger than or equal to $\text{VaR}$. The CVaR is the average losses over a $Q\%$ probability level to be identified by $\alpha$, that is losses that can be expected with this probability. The CVaR of a sample is obtained as follows:

$$CVaR_\alpha = \frac{1}{1 - \alpha} \int_{\alpha}^{1} \text{VaR}_q dq.$$  \hspace{1cm} (4.1)
To determine the efficiency of a good indicator of market risk, Artzner et al. (1997) derive four desirable properties that should comply with a measure of risk to be called "coherent". A risk indicator $\rho$ must satisfy

1. **Positive homogeneity:**
   \[
   \rho(\lambda u) = \lambda \rho(u).
   \]
   Increasing the value of portfolio in $\lambda$, the risk must also increase by $\lambda$.

2. **Monotonicity:**
   \[
   u \leq v \Rightarrow \rho(u) \leq \rho(v).
   \]
   If the portfolio $u$ has a consistently lower return than the portfolio $v$, then risk must be lower.

3. **Translation invariance:**
   \[
   \rho(u + a) = \rho(u) + a.
   \]
   Add cash of an amount $a$ then add the risk by $a$.

4. **Subadditivity:**
   \[
   \rho(u + v) \leq \rho(u) + \rho(v).
   \]
   The portfolio composition should not increase the risk.

If $\rho$ satisfies these properties, then it is considered as a coherent risk measure.
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Delta-normal method for approaches to \( VaR \)

This parametric method to calculate \( VaR \) was proposed by Baumol (1963) as a criterion of confidence limits expected gain. Assuming a confidence level fixed \( \alpha \in (0, 1] \) and a time horizon of \( T \) days, the \( VaR \) can be easily calculated from \( \sigma \), by the following expression:

\[
VaR_\alpha = \mu_S - \Phi^{-1}(\alpha) \sigma_S \sqrt{T},
\]

where \( \Phi(x) \) is the standard normal distribution function.

\[
(4.2)
\]
Traditional approaches

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Delta-normal method for approaches to CVaR

Under the assumption of normality, the parametric model that determines the CVaR of a position is as follows:

\[
CVaR_\alpha = \mu_S - \frac{\sigma_S \sqrt{T}}{1 - \alpha} \varphi (z_\alpha),
\]

where \( \varphi(x) \) is the standard normal density function. This last formula coincides with the expression given by Jondeau et al. (2009, page 335) and McNeil et al. (2005, page 45).
Based on the Fisher and Cornish (1960) expansion, Zangari (1996) approximate the percentiles of the probability distribution of \( S \) and obtain the \( \text{VaR} \) for a confidence level \( \alpha \% \) and a horizon of \( T \) days as follows

\[
\text{VaR}_\alpha = \mathbb{E}[S] - \omega_\alpha \sigma_S \sqrt{T},
\]

(4.4)

where \( \omega_\alpha \) is defined (Abramowitz and Stegun (1965)) as follows:

\[
\omega_\alpha = z_\alpha + \frac{1}{6} (z_\alpha^2 - 1) \beta_1(S) + \frac{1}{24} (z_\alpha^3 - 3z_\alpha) (\beta_2(S) - 3) \\
- \frac{1}{36} (2z_\alpha^3 - 5z_\alpha) \beta_1^2(S) - \frac{1}{24} (z_\alpha^4 - 5z_\alpha^2 + 2) \beta_1(S) (\beta_2(S) - 3),
\]

(4.5)

here \( \beta_1(S), \beta_2(S) \) denote the skewness and kurtosis from the distribution of \( S \). Note that when the skewness coefficient \( \beta_1(S) \) and excess kurtosis \( \beta_2(S) \) are zero, we obtain the quantile of the variable \( N(0, 1) \).
Cornish-Fisher approximation

The CVaR can be approximated using the Cornish-Fisher expansion for a confidence level $\alpha$% and a horizon of $T$ days as follows

$$CVaR_\alpha = \mathbb{E}[S] - \frac{1}{1 - \alpha} \omega^*_\alpha \sigma_S \sqrt{T}, \quad (4.6)$$

with

$$\omega^*_\alpha = \int_\alpha^1 \omega_q dq = \left\{ 1 + \frac{z_\alpha}{6} \beta_1(S) + \frac{z^2_\alpha - 1}{24} (\beta_2(S) - 3) - \frac{2z^2_\alpha - 1}{36} \beta^2_1(S) \right. $$

$$\left. - \frac{z^3_\alpha - 2z_\alpha}{24} \beta_1(S) (\beta_2(S) - 3) \right\} \varphi(z_\alpha),$$

where $\omega_q$ is given in (4.5). Note that when the skewness coefficient $\beta_1(S)$ and excess kurtosis $\beta_2(S)$ are zero, this expression reduces to (4.3).
Approximation by the Tukey’s $g − h$ distribution

If $S$ is approximated by $S = A + BY$, then we obtain the $VaR$ for $\alpha > 0.5$,

$$VaR_\alpha = A + BT_{g,h}(Z_\alpha)$$
$$VaR_{1-\alpha} = A - B \exp\{-gZ_\alpha\} T_{g,h}(Z_\alpha).$$

This expression coincides with the corresponding result presented in Nam and Gup (2003).
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Approximation by the Tukey’s $g - h$ distribution

If $S$ is approximated by $S = A + BY$, then we obtain the $\text{CVaR}$ for $\alpha > 0.5$,

$$\text{CVaR}_\alpha = A + B \frac{1}{1 - \alpha} \left[ \mu_{g,h} \Phi(\delta_{2\alpha}) + \frac{1}{1 - h} \frac{\Phi(\delta_{2\alpha}) - \Phi(\delta_{1\alpha})}{\delta_{2\alpha} - \delta_{1\alpha}} \right],$$

(4.8)

where $\mu_{g,h}$ is the mean of the Tukey’s $g - h$ distribution given in (2.5) and

$$\delta_{1\alpha} = -\sqrt{1 - h} z_\alpha,$$

$$\delta_{2\alpha} = \delta_{1\alpha} + \frac{g}{\sqrt{1 - h}},$$

(4.9)
When $h = 0$,

$$\text{VaR}_\alpha = A + \frac{B}{g} \left( e^{gZ_\alpha} - 1 \right) = \theta + e^{\mu + \sigma Z_\alpha}, \quad (4.10)$$

where $\theta = A - e^\mu$ and $g = \sigma$. Jiménez and Martínez (2006) propose that in this case, if $S = \ln (X - \theta)$ follows a normal law $N(\mu, \sigma^2)$, we get,

$$\mu_X = \theta + \exp \left\{ \mu + \frac{1}{2} \sigma^2 \right\} \quad \text{and} \quad \sigma^2_X = (\mu_X - \theta)^2 \left[ \exp \{\sigma^2\} - 1 \right],$$

solving for $\mu$, $\sigma$, and substituting this into (4.10) resulting

$$\text{VaR}_\alpha = \theta + \frac{\mu_X - \theta}{\sqrt{1 + \rho^2_X}} \exp \left\{ Z_\alpha \sqrt{\ln (1 + \rho^2_X)} \right\}, \quad (4.11)$$

where $\rho_X = \frac{\sigma_X}{\mu_X - \theta}$, which coincides with the coefficient of variation of random variable $X$, when $\theta = 0$. 
Suppose that \( g = 0 \), we obtain

\[
VaR_\alpha = A + B Z_\alpha e^{\frac{1}{2} h Z_\alpha^2},
\]

when \( h=1 \) we have the cauchy distribution with parameters \( \mu \) and \( \sigma \), ie

\[
VaR_\alpha = \mu + \sigma Z_\alpha e^{\frac{1}{2} Z_\alpha^2}.
\]

If \( g = h = 0 \), using the constants given for location and scale parameters in Jiménez (2004), we obtain

\[
VaR_\alpha = \mu + \sigma Z_\alpha.
\]

Note that this last expression coincides with the classical formula of VaR (see Jorion (2007)).
Supposing that $h = 0$, then

$$CVaR_{\alpha} = \theta + \frac{\Phi (\sigma - z_{\alpha})}{1 - \alpha} \ e^{\mu + \sigma^2/2}, \quad (4.12)$$

where $\theta = A - e^\mu$. Following the approach used in Jiménez (2004), by solving $\mu, \sigma$, to replace (4.12), it follows that

$$CVaR_{\alpha} = \theta + \frac{\mu X - \theta}{1 - \alpha} \left\{ \Phi \left[ \sqrt{\ln (1 + \rho_X^2)} - Z_{\alpha} \right] \right\}, \quad (4.13)$$

where $\rho_X = \frac{\sigma_X}{\mu_X - \theta}$, which coincides with the coefficient of variation of random variable $X$, when $\theta = 0$. 

Traditional approaches

1. If $g = 0$, we can use the Mean Value Theorem,

$$
\frac{\Phi(b) - \Phi(a)}{b - a} \approx \varphi(c), \quad \text{where} \quad c \in (a, b),
$$

accordingly we obtain

$$
CVaR_\alpha = A + \frac{B}{1 - \alpha} \frac{\varphi(\sqrt{1 - h} Z_\alpha)}{1 - h}.
$$

2. When $g = h = 0$, using the constants given for location and scale parameters in Jiménez (2004), we have

$$
CVaR_\alpha = \mu - \frac{\sigma}{1 - \alpha} \varphi(Z_\alpha).
$$

Note that this last expression coincides with the formula for the $CVaR$ given in (4.3).
An Illustration

Now we compare the procedure developed above with the classical method, historical simulation and the Cornish-Fisher approximation to estimate the VaR and CVaR.

We consider a portfolio constructed with three largest market capitalization stocks in Spain: Banco Bilbao Vizcaya Argentaria (BBVA), Endesa (ELE) and Banco Santander (SAN). The data were obtained from http://es.finance.yahoo.com and the sample covers 2,081 trading days from 01/01/2003 to 17/01/2011.

1. Get the arithmetic daily rate of return for each stock, ie

\[ R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \quad t = 1, 2, \ldots, T, \tag{5.1} \]

where \( P_t \) denotes the price of the stock at time \( t \).

<table>
<thead>
<tr>
<th>Asset</th>
<th>Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>BBVA (%)</td>
<td>0.0163312757</td>
</tr>
<tr>
<td>ELE (%)</td>
<td>0.0424676054</td>
</tr>
<tr>
<td>SAN (%)</td>
<td>0.0351851054</td>
</tr>
</tbody>
</table>
Covariance matrix of the portfolio

\[
\Sigma = \begin{bmatrix}
4,325674189 & 1,714872578 & 4,100211925 \\
1,714872578 & 2,961638820 & 1,738771858 \\
4,100211925 & 1,738771858 & 4,677008713
\end{bmatrix}
\]

The Global Minimum Variance Portfolio (GMVP)

<table>
<thead>
<tr>
<th>Asset</th>
<th>GMVP</th>
</tr>
</thead>
<tbody>
<tr>
<td>BBVA</td>
<td>0.25574446</td>
</tr>
<tr>
<td>ELE</td>
<td>0.67213809</td>
</tr>
<tr>
<td>SAN</td>
<td>0.072117444</td>
</tr>
</tbody>
</table>

We assume an investment of \( V_0 = 1 \) million currency units and positions in this portfolio is

<table>
<thead>
<tr>
<th>Asset</th>
<th>GMVP</th>
</tr>
</thead>
<tbody>
<tr>
<td>BBVA</td>
<td>255,744</td>
</tr>
<tr>
<td>ELE</td>
<td>672,138</td>
</tr>
<tr>
<td>SAN</td>
<td>72,117</td>
</tr>
</tbody>
</table>
Now \( \mu_V = w'r = 352,58188 \) and \( \sigma^2_V = 2,55459 \times 10^8 \). If \( X \) is approximated by \( X = A + BY \), then

\[
X = 243,9427 - 11,968,1342 \frac{1}{g} \left[ \exp\{gZ\} - 1 \right] \exp \left\{ \frac{h}{2} Z^2 \right\};
\]

where

\[
g = -0,29507 \quad \text{and} \quad h = 0,11718.
\]

As shown in figure 1, there is a difference between the empirical distribution of portfolio returns GMVP (represented by the histogram) and the normal distribution. Tukey's \( g - h \) distribution better approximates the empirical. The distribution of portfolio returns as expected GMVP tends to be more leptokurtic than the normal distribution has heavier tails.
Figura: Portfolio vs. Normal Distribution and Tukey’s $g - h$ distribution
The following table presents the statistics of portfolio returns GMVP.

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>352.58188</td>
</tr>
<tr>
<td>Median</td>
<td>270.60724</td>
</tr>
<tr>
<td>Stan. Dev.</td>
<td>15,983.10112</td>
</tr>
<tr>
<td>Minimum</td>
<td>-174,798.6684</td>
</tr>
<tr>
<td>Maximum</td>
<td>121,477.92095</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.33695</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>15.80073</td>
</tr>
<tr>
<td>JB test</td>
<td>14,240.4450</td>
</tr>
</tbody>
</table>

**Cuadro:** Descriptive Statistics

Kurtosis, skewness and the test proposed by Jarque and Bera (1987), statistics reported in Table 2 indicate that the null hypothesis of normal distribution can be rejected for the variable under study. Figure 1 shows such a histogram, it is evident that the returns of the series have a slight degree of bias to the right, leptokurtic and do not follow the normal distribution.
In this case, the calculation of VaR, initially assumed to be normal in returns. Table 3 presents the loss of calculate VaR for the portfolio GMVP under the following confidence levels: 90 %, 95 %, 97.5 % and 99 %.

<table>
<thead>
<tr>
<th>VaR</th>
<th>$\alpha = 0.90$</th>
<th>$\alpha = 0.95$</th>
<th>$\alpha = 0.975$</th>
<th>$\alpha = 0.99$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classical</td>
<td>20,130</td>
<td>25,937</td>
<td>30,973</td>
<td>36,829</td>
</tr>
<tr>
<td>Historical</td>
<td>15,279</td>
<td>22,262</td>
<td>28,423</td>
<td>48,198</td>
</tr>
<tr>
<td>Cornish-Fisher</td>
<td>19,553</td>
<td>24,406</td>
<td>48,381</td>
<td>32,869</td>
</tr>
<tr>
<td>GH ($A, B, g, h$)</td>
<td>20,279</td>
<td>29,448</td>
<td>39,533</td>
<td>54,704</td>
</tr>
</tbody>
</table>

Cuadro: Comparison of VaR methodologies
As shown in Figure 2 there is a perceptible difference between the VaR methodologies.
Table 4 presents the loss of calculate CVaR for the portfolio GMVP under the following confidence levels: 90 %, 95 %, 97.5 % and 99 %.

<table>
<thead>
<tr>
<th>CVaR</th>
<th>Confidence levels</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha = 0.90$</td>
</tr>
<tr>
<td>Classical</td>
<td>27,697</td>
</tr>
<tr>
<td>Cornish-Fisher</td>
<td>35,087</td>
</tr>
<tr>
<td>GH ($A, B, g, h$)</td>
<td>41,660</td>
</tr>
</tbody>
</table>

**Cuadro:** Comparison of CVaR methodologies

As can be noted CVaR losses for each of the methods are greater than the losses of VaR.
Figure 3 shows that there is a difference between the CVaR methodologies.

Figura: Comparison of CVaR methodologies
Conclusions

This article presents an alternative methodology to establish the \textit{VaR} and \textit{CVaR} when the portfolio distribution has skewness and kurtosis. Whereas, normality assumption overestimates the \textit{VaR} and \textit{CVaR} for upper percentiles, while it underestimates for lower percentiles of values that correspond to extreme values. The formulas obtained to calculate \textit{VaR} and \textit{CVaR} are explicit and we get the classical model as a particular case when the parameters $g$ and $h$ are considered equal to zero. The losses obtained for \textit{CVaR} for each of the methods used are greater than losses of \textit{VaR}. Thus our model exhibiting many of the characteristics of the other models in the literature.
References I


