General Methods for system Reliability Evaluation

- Reliability - Most important factor in Eng system design
- Laboratory testing
- Complex system
- System reliability in terms of component reliability
- Depends on the member of components and structure

Parallel and Series Reductions
- An easy method if it works
- Replace a series or parallel subsystem with a super component
Example: Consider a system whose reliability block diagram is given in Figure a. The system has seven components. The reliability and unreliability of component $i$ are given and denoted by $p_i$ and $q_i$, respectively ($i = 1, 2, \ldots, 7$). Use series and parallel reductions for system reliability evaluation.

From the system structure given in Figure a, we see that components 3 and 4 form a series subsystem, which can be represented by a “supercomponent” denoted by 3–4. The reliability of supercomponent 3–4 is equal to the product of the reliabilities of components 3 and 4:

$$p_{3–4} = p_3 p_4.$$ 

Components 5, 6, and 7 also form a series subsystem, which can be represented by a supercomponent denoted by 5–6–7. The reliability of supercomponent 5–6–7 is equal to the product of the reliabilities of components 5, 6, and 7:

$$p_{5–6–7} = p_5 p_6 p_7.$$ 

After these two series reductions, the reliability block diagram in Figure a is transformed into that in Figure b.

Examination of Figure b reveals that component 2 and component 3–4 form a parallel subsystem denoted by supercomponent 2–3–4, whose unreliability is the product of the unreliabilities of component 2 and supercomponent 3–4:

$$q_{2–3–4} = q_2 q_{3–4} = q_2 (1 - p_{3–4}) = q_2 (1 - p_3 p_4).$$
After this parallel reduction, the reliability block diagram in Figure 1b is transformed into that in Figure 1c.

Figure 1c shows that component 1 and supercomponent 2–3–4 form a series subsystem and a series reduction produces a supercomponent denoted by 1–2–3–4, whose reliability is

\[ P_{1-2-3-4} = p_1 p_{2-3-4} = p_1 (1 - q_{2-3-4}) = p_1 [1 - q_2 (1 - p_3 p_4)]. \]

This series reduction transforms Figure 1c to Figure 1d.

Figure 1d can be further simplified with a parallel reduction to generate Figure 1e, which has only one supercomponent, denoted by 1–2–3–4–5–6–7. The unreliability of this supercomponent is

\[ q_{1-2-3-4-5-6-7} = q_{1-2-3-4} q_{5-6-7} = [1 - p_1 (1 - q_2 (1 - p_3 p_4))](1 - p_5 p_6 p_7). \]

The system reliability is equal to the reliability of the final supercomponent in Figure 1e:

\[ R_s = 1 - q_{1-2-3-4-5-6-7} = 1 - [1 - p_1 (1 - q_2 (1 - p_3 p_4))](1 - p_5 p_6 p_7). \]
Using series and parallel reductions in Example
usually unable to simplify a general network into a single supercomponent. Other techniques to be discussed later will be applied after parallel and series reductions to find the exact system reliability. The following example shows that parallel and series reductions can simplify a system structure but not to the point of finding exact system reliability.
Example Consider the network depicted in Figure a. The nine failure-prone components are numbered from 1 to 9. We are interested in finding the reliability for the source to be able to communicate to the sink.

Examining Figure a, we can see that series reductions can be applied to components 1 and 2 resulting in a supercomponent called 1–2 and to components 3 and 4 resulting in a supercomponent called 3–4. The reliabilities of these two supercomponents are

\[ p_{1-2} = p_1 p_2, \quad p_{3-4} = p_3 p_4. \]

A parallel reduction can be applied to components 7 and 8 resulting in a supercomponent 7–8 whose unreliability is given by

\[ q_{7-8} = q_7 q_8. \]

After these series and parallel reductions, the network diagram in Figure a is transformed into the network diagram in Figure b.

Applying a series reduction to component 6 and supercomponent 7–8 transforms Figure b to Figure c with

\[ p_{6-7-8} = p_6 (1 - q_{7-8}) = p_6 (1 - q_7 q_8). \]

No more series or parallel reductions can be used to further simplify the network diagram in Figure c. We have simplified the original network diagram into a bridge structure.
PIVOTAL DECOMPOSITION

The pivotal decomposition method is based on the concept of conditional probability. The following equation illustrates the idea behind this method:

\[
\Pr(\text{system works}) = \Pr(\text{component } i \text{ works}) \Pr(\text{system works } | \text{ component } i \text{ works}) + \Pr(\text{component } i \text{ fails}) \Pr(\text{system works } | \text{ component } i \text{ fails}).
\]

The efficiency of this method depends on the ease of evaluating the conditional probabilities. This means that the selection of the component to be decomposed may play an important role in the efficiency of this method. If the decomposition of a selected component results in two system structures for which parallel and/or series reductions can be applied again, the efficiency of the system reliability evaluation will be enhanced. We will use an example to illustrate this method.

Example

Consider the bridge structure. No parallel or series reductions may be applied to the bridge structure directly. There are five components in Figure 4.6. A decomposition on any component will result in system structures to which parallel and series reductions can be applied.

In Example 4.9, we selected component 3 to be decomposed in derivation of the structure and logic functions of the bridge structure. In this example, again, we will choose component 3 to be decomposed first. Using equation (5.1), we have

\[
R_S = p_3 \Pr(\text{system works } | \text{ component } 3 \text{ works}) + q_3 \Pr(\text{system works } | \text{ component } 3 \text{ fails}).
\]
The reliability block diagram of the bridge structure, under the condition that component 3 works, is given in Figure 4.8. Applying parallel and series reductions to the system in Figure 4.8 results in

\[ \Pr(\text{system works} \mid \text{component 3 works}) = (1 - q_1 q_4)(1 - q_2 q_5). \]

The reliability block diagram of the bridge structure, under the condition that component 3 fails, is given in Figure 4.8. Applying parallel and series reductions to the system in Figure 4.8 results in

\[ \Pr(\text{system works} \mid \text{component 3 fails}) = 1 - (1 - p_1 p_2)(1 - p_4 p_5). \]

Substituting these two conditional probabilities into equation (5.2) yields the reliability of the bridge structure:

\[ R_s = p_3(1 - q_1 q_4)(1 - q_2 q_5) - q_3[1 - (1 - p_1 p_2)(1 - p_4 p_5)]. \]

The bridge structure is a commonly seen system reliability structure. It either exists by itself as a subsystem in a system or appears when other network transformation techniques have been applied. In Example 5.2, parallel and series reductions transform the original system structure into a bridge structure. Since the reliability of a bridge structure is derived in Example 5.3, we can standardize its expression so that it can be used whenever a bridge structure is spotted. The standard bridge structure used in this book is defined as follows,
- The reliability block diagram of a bridge structure is given in Figure 1. Figure may also represent a network diagram with a failure-prone link. The diagram has the shape of a diamond. Communications, or signal flows, are from left to right.
- Component 3 is the center of the bridge structure.
- The two components on the top two edges are numbered as 1 and 2 from left to right. The two components on the bottom two edges are numbered as 4 and 5 from left to right.

With these specifications for a standard bridge structure, its system reliability expression is given in equation (5.3).

**Example.** Consider the network diagram given in Figure 1a. Links are failure prone while the nodes are perfect. We are interested in the reliability of communications between the source node and the sink node.

There are seven links in this network. We will pick component 2 (link 2) as the one to be decomposed. When component 2 is perfect, the original network diagram in Figure 1a is simplified to that in Figure 1b. Parallel and series reductions can be applied to find the reliability of the two-terminal network diagram in Figure 1b:

\[
\Pr(\text{system works | component 2 works}) = (1 - q_1q_5)(1 - q_4[1 - p_7(1 - q_3q_6)]).
\]
FIGURE Use of decomposition in two-terminal network reliability evaluation.
When component 2 is failed, the original network diagram in Figure $a$ is simplified to that in Figure $c$. A series reduction on components 5 and 6 produces a supercomponent called 5–6 in the place of link 5 and link 6. Mapping the components in Figure $c$ to the standard bridge structure given in Figure and applying equation , we find the two-terminal reliability of the network in Figure $c$ as follows:

\[
\Pr(\text{system works} \mid \text{component 2 fails}) = p_3(1 - q_1 q_{5-6})(1 - q_4 q_7) \\
+ q_3[1 - (1 - p_1 p_4)(1 - p_{5-6} p_7)] \\
= p_3[1 - q_1 (1 - p_5 p_6)](1 - q_4 q_7) \\
+ q_3[1 - (1 - p_1 p_4)(1 - p_{5} p_{6} p_7)].
\]

Substituting these two conditional probabilities into equation , we obtain the reliability of the network depicted in Figure $a$:
\[ R_s = p_2 \Pr(\text{system works \mid component 2 works}) \]
\[ + q_2 \Pr(\text{system works \mid component 2 fails}) \]
\[ = p_2(1 - q_1 q_5)(1 - q_4[1 - p_7(1 - q_3 q_6)]) \]
\[ + q_2\{p_3[1 - q_1(1 - p_5 p_6)](1 - q_4 q_7) \]
\[ + q_3[1 - (1 - p_1 p_4)(1 - p_5 p_6 p_7)]\}. \]
Inclusion–exclusion (IE) is a classical method for producing the reliability expression of a general system using its minimal paths or minimal cuts. The IE method, also known as Poincaré and Sylvester’s theorem, provides successive upper and lower bounds by Bonferroni inequalities on system reliability that converge to the exact system reliability.

Let $E_j$ be the event that all components in the minimal path $T_j$ work. We also say that $E_j$ represents the event that minimal path $T_j$ works. The probability that the minimal path $T_j$ works can be expressed as

$$\Pr(E_j) = \prod_{i \in T_j} p_i.$$ 

A system with $l$ minimal paths works if and only if at least one of the minimal paths works. In other words, system success corresponds to the event $\bigcup_{j=1}^{l} E_j$. The reliability of the system is equal to the probability of the union of $l$ events, namely,

$$R_s = \Pr \left( \bigcup_{j=1}^{l} E_j \right).$$
Let

\[ S_k = \sum_{1 \leq i_1 < \ldots < i_k \leq l} \Pr \left( E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k} \right). \]

Then, \( S_k \) represents the sum of the probabilities that any \( k \) minimal paths are simultaneously working. By the IE principle (see Feller [75]), the reliability of the system, which is equal to the probability of the union of the \( l \) minimal paths, can be expressed as

\[ R_s = \sum_{k=1}^{l} (-1)^{k-1} S_k. \]

In application of equation (5.13), \( S_1 \) is included, \( S_2 \) is excluded, \( S_3 \) is included, \( S_4 \) is excluded, and so on. This is where the name of the IE method comes from. In this process of including and excluding additional terms, upper and lower bounds on \( R_s \) become available, as given below:
\[ R_s \leq S_1, \]
\[ R_s \geq S_1 - S_2, \]
\[ R_s \leq S_1 - S_2 + S_3, \]
\[ R_s \geq S_1 - S_2 + S_3 - S_2. \]
\[ R_s \leq S_1 - S_2 + S_3 - S_2 - S_5. \]

These inequalities are the so-called Bonferroni inequalities. Tighter bounds on \( R_s \) are provided by these successive inequalities and, eventually, the exact value of \( R_s \) is obtained when \((-1)^{l-1} S_l\) is included. In practice, it may be necessary to calculate only the first few \( S_k \) values in order to obtain an \( R_s \) value that is regarded as accurate.

In a parallel system with \( n \) components, there are \( n \) minimal paths. Because these \( n \) minimal paths do not have components in common, equation (5.13) has the maximum number of possible terms, \( 2^n - 1 \). This is because \( S_1 \) has \( \binom{n}{1} \) terms, \( S_2 \) has \( \binom{n}{2} \) terms, \( \ldots \), \( S_n \) has \( \binom{n}{n} \) terms. Apparently, the IE method is not as efficient method for reliability evaluation of a parallel system. The reliability of a parallel system can be easily evaluated with the simple formula in equation
For a system with \( l \) minimal paths, the maximum number of possible terms generated by the IE method is \( 2^l - 1 \). This would occur only when there is no component in common for any two minimal paths. Usually, some minimal paths would have components in common. For example, for the network diagram given in Figure 5.4 covered in the previous section, component 1 appears in three different minimal paths, namely MP\(_1\), MP\(_3\), and MP\(_5\). Whenever there are common components in some minimal paths, some of the \( 2^l - 1 \) possible terms of the IE method cancel each other because of the alternating signs in front of \( S_k \) for \( 1 \leq k \leq l \). As a result, the actual number of final terms generated by the IE method is usually much smaller than \( 2^l - 1 \). However, the IE method has to evaluate all these \( 2^l - 1 \) terms and then let some of the terms cancel each other to produce the final result. In other words, this method is not a very efficient method for systems with a large number of minimal paths.
Example Use the IE method to find the reliability of the bridge network given in Figure 4.6. The minimal paths given in Example 4.6 are

\[ T_1 = \{1, 2\}, \quad T_2 = \{4, 5\}, \quad T_3 = \{1, 3, 5\}, \quad T_4 = \{2, 3, 4\}. \]

The logic functions of the minimal paths denote the events that each minimal path works and are given in Example 4.7. We will use \( MP_i \) to indicate the logic function of the \( i \)th minimal path. Then, we have

\[ MP_1 = x_2x_2, \quad MP_2 = x_4x_5, \quad MP_3 = x_1x_3x_5, \quad MP_4 = x_2x_3x_4. \]

Applying the IE method, we have
\[ S_1 = \Pr(\text{MP}_1) + \Pr(\text{MP}_2) + \Pr(\text{MP}_3) + \Pr(\text{MP}_4) = p_1p_2 + p_4p_5 + p_1p_3p_5 + p_2p_3p_4, \]

\[ S_2 = \Pr(\text{MP}_1\text{MP}_2) + \Pr(\text{MP}_1\text{MP}_3) + \Pr(\text{MP}_1\text{MP}_4) + \Pr(\text{MP}_2\text{MP}_3) + \Pr(\text{MP}_2\text{MP}_4) \]

+ \Pr(\text{MP}_3\text{MP}_4).

\[ = p_1p_2p_4p_5 + p_1p_2p_3p_5 + p_1p_2p_3p_4 + p_1p_3p_4p_5 + p_2p_3p_4p_5 + p_1p_2p_3p_4p_5, \]

\[ S_3 = \Pr(\text{MP}_1\text{MP}_2\text{MP}_3) + \Pr(\text{MP}_2\text{MP}_2\text{MP}_4) + \Pr(\text{MP}_1\text{MP}_3\text{MP}_4) + \Pr(\text{MP}_2\text{MP}_3\text{MP}_4) \]

\[ = 4p_1p_2p_3p_4p_5. \]

\[ S_4 = \Pr(\text{MP}_1\text{MP}_2\text{MP}_3\text{MP}_4) = p_1p_2p_3p_4p_5. \]

\[ R_s = S_1 - S_2 + S_3 - S_4 \]

\[ = p_1p_2 + p_4p_5 + p_1p_3p_5 + p_2p_3p_4 \]

\[ - (p_1p_2p_5 + p_1p_2p_3p_5 + p_1p_2p_3p_4 + p_1p_3p_4p_5 + p_1p_3p_4p_5) + 2p_1p_2p_3p_4p_5. \]
If all components are i.i.d. with reliability $p$, the reliability of the bridge structure becomes

$$R_s = 2p^2 + 2p^3 - 5p^4 + 2p^5.$$  

In this example, there are four minimal paths, that is, $l = 4$. In evaluating $S_k$ for $1 \leq k \leq 4$, $2^4 - 1 = 15$ terms are evaluated. In the final expression of $R_s$, 11 terms are added or subtracted. This means that four terms are canceled.

For reliability evaluation of the bridge structure, the IE method is not as efficient as the decomposition method in combination with parallel and series reductions illustrated through Example.

The IE method is based on the IE principle for evaluation of the union of several events. We have illustrated its use in system reliability evaluation using minimal paths. It can also be used for system unreliability evaluation using minimal cuts.

Let $E_j$ be the event that all components in minimal cut set $C_j$ fail. We also say that $E_j$ represents the event that minimal cut $C_j$ fails. The probability that the minimal cut $C_j$ fails can be expressed as

$$\Pr(E_j) = \prod_{i \in C_j} q_i.$$
A system with \( m \) minimal cuts fails if and only if at least one of the minimal cuts fails. In other words, system failure corresponds to the event \( \bigcup_{j=1}^{m} \overline{E}_j \). The unreliability of the system is equal to the probability of the union of the \( m \) events, namely,

\[
Q_s = \Pr \left( \bigcup_{j=1}^{m} \overline{E}_j \right).
\]
Let
\[ V_k = \sum_{1 \leq i_1 < \ldots < i_k \leq m} \Pr(\overline{E}_{i_1} \cap \overline{E}_{i_2} \cap \cdots \cap \overline{E}_{i_k}). \]

Then, \( V_k \) represents the sum of the probabilities that any \( k \) of the minimal cuts are simultaneously failed. The system unreliability, which is equal to the probability of the union of the failures of the \( m \) minimal cuts, can be expressed as
\[ Q_s = \sum_{k=1}^{m} (-1)^{k-1} V_k. \]

In application of equation, \( V_1 \) is included, \( V_2 \) is excluded, \( V_3 \) is included, \( V_4 \) is excluded, and so on. In this process of including and excluding additional terms, upper and lower bounds on \( Q_s \) become available, as given below:

\[
\begin{align*}
Q_s &\leq V_1, \\
Q_s &\geq V_1 - V_2, \\
Q_s &\leq V_1 - V_2 + V_3, \\
Q_s &\geq V_1 - V_2 + V_3 - V_4, \\
Q_s &\leq V_1 - V_2 + V_3 - V_4 + V_5, \\
& \vdots
\end{align*}
\]
Example. Consider the bridge structure given in Figure. The minimal cuts given in Example 4.6 are

\[ C_1 = \{1, 4\}, \quad C_2 = \{2, 5\}, \quad C_3 = \{1, 3, 5\}, \quad C_4 = \{2, 3, 4\}. \]

The logic functions of the minimal cuts denote the events that each minimal cut fails and are given in Example. We use \( \text{MC}_i \) to indicate the logic function of the \( i \)th minimal cut. Then, we have

\[ \text{MC}_1 = \overline{x}_1 \overline{x}_4, \quad \text{MC}_2 = \overline{x}_2 \overline{x}_5, \quad \text{MC}_3 = \overline{x}_1 \overline{x}_3 \overline{x}_5, \quad \text{MC}_4 = \overline{x}_2 \overline{x}_3 \overline{x}_4. \]

Applying the IE method, we have

\[ V_1 = \Pr(\text{MC}_1) + \Pr(\text{MC}_2) + \Pr(\text{MC}_3) + \Pr(\text{MC}_4) \]
\[ = q_1 q_4 + q_2 q_5 + q_1 q_3 q_5 + q_2 q_3 q_4, \]

\[ V_2 = \Pr(\text{MC}_1 \text{MC}_2) + \Pr(\text{MC}_1 \text{MC}_3) + \Pr(\text{MC}_1 \text{MC}_4) + \Pr(\text{MC}_2 \text{MC}_3) \]
\[ + \Pr(\text{MC}_2 \text{MC}_4) + \Pr(\text{MC}_3 \text{MC}_4) \]
\[ = q_1 q_2 q_4 q_5 + q_1 q_3 q_4 q_5 + q_1 q_2 q_3 q_4 + q_1 q_2 q_3 q_5 + q_2 q_3 q_4 q_5 + q_1 q_2 q_3 q_4 q_5. \]
\[ V_3 = \Pr(MC_1MC_2MC_3) + \Pr(MC_1MC_2MC_4) \\
+ \Pr(MC_1MC_3MC_4) + \Pr(MC_2MC_3MC_4) \\
= 4q_1q_2q_3q_4q_5, \]

\[ V_4 = \Pr(MC_1MC_2MC_3MC_4) \\
= q_1q_2q_3q_4q_5, \]

\[ R_s = 1 - (V_1 - V_2 + V_3 - V_4) \\
= 1 - q_1q_4 - q_2q_5 - q_1q_3q_5 - q_2q_3q_4 + q_1q_2q_4q_5 + q_1q_3q_4q_5 + q_1q_2q_3q_4 \\
+ q_1q_2q_3q_5 + q_2q_3q_4q_5 - 2q_1q_2q_3q_4q_5. \]

When all components are i.i.d. with unreliability \( q \), the unreliability of the bridge structure becomes

\[ Q_s = 2q^5 - 5q^4 + 2q^3 + 2q^2. \]
SUM-OF-DISJOINT-PRODUCTS METHOD

Fratta and Montanari [77] first reported the sum-of-disjoint-products (SDP) method in 1973. Abraham [2] published an improved version of SDP. Many other authors provided improvements on this method. Locks would be the conspicuous author who made the most of SDP in system reliability evaluation [146]–[149].

The SDP method uses minimal paths or minimal cuts to evaluate the probability of the union of several events. The union of the minimal paths can be expressed by the logic function of the system. This logic function can be expressed as a union of several terms. If these terms of the logic function are disjoint, then there is a one-to-one correspondence between this expression of the logic function of the system and the reliability measure of the system. Thus, the focus of the method is on expressing the logic function of the system as a union of disjoint terms. Each of these disjoint terms is a product of the events that individual components work or fail. The SDP method differs from the IE method in the signs (plus or minus) of the terms in the system reliability formula. With the IE method, the signs of the terms alternate between plus and minus with the plus signs denoting the sets whose probability is added in the reliability formula and the minus signs denoting the sets whose reliability is subtracted from the reliability formula due to double counting in the previous inclusion operations. With the SDP method, however, all sets are included, none are double counted, and all terms have the plus signs. An SDP formula for any but very small systems is smaller than the IE formula [149].

Disjoint Terms: Addition Law The addition law of probabilities is the underlying justification for the SDP method. If two or more events have no elements in common, the probability that at least one of the events will occur is the sum of the probabilities...
Other popular methods are:
MCIS Technique
Sum of disjoint product Method
Delta – Star and Star – Delta Transformations
Esary – Proschan Method
Reliability is one of the most important considerations in engineering system design today. In optimal reliability design problems, the objective function may be maximization of system reliability, maximization of system availability, minimization of cost, or minimization of system life-cycle cost. The constraints may include budget restrictions, reliability requirements, and other considerations such as volume and weight. The design parameters or decision variables may be component reliability value, number of redundancies, or arrangement of known components.

Reliability design problems can be formulated as optimization problems, and various optimization algorithms have been used to solve them (Tillman et al., Kuo and Prasad, and Kuo et al.). The major focus of recent work in the area of system reliability optimization is on the development of heuristic methods and metaheuristic algorithms for redundancy allocation problems. Little work is directed toward exact solutions for such problems. To the best of our knowledge, all of the reliability systems considered in this area belong to the class of coherent systems. The literature on reliability optimization methods can be classified into seven categories.
1. Heuristics for redundancy allocation: special techniques developed for reliability problems.

2. Metaheuristic algorithms for redundancy allocation: perhaps the most attractive development in the 1990s.

3. Exact algorithms for redundancy allocation or reliability allocation: most are based on mathematical programming techniques, for example, the reduced gradient methods.


5. Multiobjective system reliability optimization: an important but not widely studied problem in reliability optimization.

6. Optimal assignment of interchangeable components: a unique scheme that often takes no effort.

7. Others: including decomposition, fuzzy apportionment, and effort function minimization.
A special class of reliability design problems is the allocation of reliability values to various components. Assignment algorithms have been used to solve this kind of problem. Because of the special characteristics of this class of problems, many interesting results have been reported.

**REdundancy in System Design**

The most commonly used structures in reliability systems are the series and the parallel structures. Each of the components in a series system is essential for the function of the whole system. For example, an automobile may be modeled as a series system with the following four major components: engine, transmission, steering, and braking. A computer system may be considered to be a series system with the following major components: CPU, motherboard, hard disk, disk controller, display card, monitor, keyboard, and mouse. The number of components needed in a series system depends on the function of the system. The function of each component is an essential part of the function of the whole system.
The reliability of a series system is very much affected by the number of components in the system. The more components a series system has, the lower the reliability of the system is. Figure a shows the reliability of a series system with i.i.d. component reliability \( p = 0.9 \) as a function of the number of components \( n \) in the system. It can be seen from Figure a that the reliability of the series system decreases as \( n \) increases. Even when the reliability of each component is 0.9, the reliability of the series system becomes only about 0.2 when there are 15 such com-
FIGURE  System reliability as a function of (a) system size $n$ and (b) i.i.d. component reliability $p$. 
ponents in the system. As a result, significant effort has been made by researchers and designers to reduce the number of components that are connected in series. Integration has been used to combine the functions of several components. For example, the reliability of VCRs has been increased significantly because less moving parts are used in today’s designs.

The reliability of a series system is smaller than that of the weakest component. When the components are i.i.d. with component reliability \( p \), system reliability is smaller than \( p \). Figure 6 shows the reliability of a five-component series system as a function of the i.i.d. component reliability \( p \). The system reliability is very close to zero when \( p \) is less than 0.6. For the system reliability to be close to 1, the component reliability must be much closer to 1. With new materials, advanced manufacturing technologies, and new designs, the reliabilities of components have been increasing steadily. However, after a certain limit, further increase of component reliability becomes very costly. An alternative method of increasing the reliability of a series system is to apply redundancy at the component level. This will produce parallel subsystems.
A parallel system structure requires at least one component to work. As long as one component is working, the system is working. All other components are called redundant components. System reliability is higher than that of the best component. The more components a parallel system has, the higher the system reliability. Figure a shows the reliability of a parallel system with i.i.d. component reliability $p = 0.5$ as a function of the number of components in the system. It can be seen that the system reliability is virtually equal to 1 when $n$ reaches 9, even when the component reliability is only 0.5. Actually, no matter how low the component reliability is, we can always achieve very high system reliability through redundancy. However, there is a diminishing return from each additional component as more components are used. In addition, there may be other concerns such as weight, volume, and costs that prevent us from using too many redundant components. Figure b shows the reliability of a five-component parallel system as a function of the i.i.d. component reliability $p$. From this figure, we see that the reliability of the parallel system is above 0.9 even when the component reliability is only about 0.4 and is virtually 1 when $p$ is about 0.7.
Redundancy is the most effective when applied at the lowest level in a hierarchical system. Consider a series system that has two i.i.d. components labeled 1 and 2. The reliability of each component is 0.9. The reliability of the series system with no redundancy is equal to 0.81. To increase the reliability of the system, we compare the following two alternative designs. Both designs have four i.i.d. components. Design 1 applies redundancy at the component level; namely, each component is allocated a redundancy. Design 1 is shown in Figure (a). The reliability of the system following design 1 is equal to 0.9801. Design 2 applies redundancy at the subsystem level. The original system with two components in series is considered to be a subsystem. Another identical subsystem with two identical components connected in series is used as a redundant subsystem in design 2. Design 2 is shown in Figure (b). The reliability of the system following design 2 is equal to 0.9639. This illustrates that whenever possible, redundancy should be applied at the lowest level in a hierarchical system. Kuo et al. provide a mathematical proof for this result.

![Diagram](image.png)

**FIGURE** Different philosophies of using redundancy.
The K out of n System Model

- An n comp system that works only if at least k of the n comp work.
- Series system – n out of n
- Parallel system – 1 out of n
- Redundancy - appl in industry, military.
- A very popular redundancy type in fault – tolerant systems.
- Multi display in a cockpit.
- Multi engine in airplane
- Multi pump system in hydraulic control system.
- Car with Vs engine, needs only 4 engines to fire.
- VLSI
In a $k$-out-of-$n$ system with i.i.d. components, the number of working components follows the binomial distribution with parameters $n$ and $p$. Thus, we have

$$\Pr(\text{exactly } i \text{ components work}) = \binom{n}{i} p^i q^{n-i}, \quad i = 0, 1, 2, \ldots, n.$$  

The reliability of the system is equal to the probability that the number of working components is greater than or equal to $k$:

$$R(k, n) = \sum_{i=k}^{n} \binom{n}{i} p^i q^{n-i}.$$

Equation is an explicit formula that can be used for reliability evaluation of the $k$-out-of-$n$ system.

If we apply the pivotal decomposition to component $n$ the system reliability of a $k$-out-of-$n$ system with i.i.d. components can be expressed as
\[ R(k, n) = pR(k-1, n-1) + (1-p)R(k, n-1) \]
\[ = p(R(k-1, n-1) - R(k, n-1)) + R(k, n-1) \]
\[ = p \Pr(\text{exactly } k-1 \text{ out of } n-1 \text{ components work}) + R(k, n-1) \]
\[ = \binom{n-1}{k-1} p^k q^{n-k} + R(k, n-1). \]

Rearranging the terms in equation, we obtain the expression
\[ R(k, n) - R(k, n-1) = \binom{n-1}{k-1} p^k q^{n-k} \quad \text{for } n \geq k. \]

Equation represents the improvement in system reliability by increasing the number of components in the system from \( n-1 \) to \( n \). As \( n \) increases, this improvement amount in system reliability will become smaller. Thus, there is an optimal design issue of determining the system size \( n \), which will be addressed later.

Equation can be used recursively for system reliability evaluation with the boundary condition
\[ R(k, n) = 0 \quad \text{for } n < k. \]
Using equation and the boundary condition given in equation, we can express the reliability of a \( k \)-out-of-\( n \):G system as follows:

\[
R(k, n) = \sum_{i=k}^{n} [R(k, i) - R(k, i - 1)] = p^k \sum_{i=k}^{n} \binom{i - 1}{k - 1} q^{i-k}.
\]

From the equations for system reliability given above, we can see that the reliability of a \( k \)-out-of-\( n \):G system with i.i.d. components is a function of \( n, k, \) and \( p \).

An increase in \( n \) or \( p \) or both or a decrease in \( k \) will increase the system's reliability. Equation represents the increase in system reliability by increasing the number of components in the system from \( n - 1 \) to \( n \). In the following, we give an expression for the increase in system reliability for each unit of decrease in \( k \):
\[ R(k, n) = \Pr(\text{at least } k \text{ components work}) \]
\[ = \Pr(\text{at least } k - 1 \text{ components work}) - \Pr(\text{exactly } k - 1 \text{ components work}) \]
\[ = R(k - 1, n) - \binom{n}{k - 1} p^{k-1} q^{n-k+1}. \]

Or equivalently, we have
\[ R(k - 1, n) - R(k, n) = \binom{n}{k - 1} p^{k-1} q^{n-k+1}. \]
With the various expressions of $R(k, n)$ derived so far, we can easily find the expressions of the unreliability $Q(k, n)$ of the $k$-out-of-$n$:G system. For example, the following is obvious from equation:

$$Q(k, n) = 1 - R(k, n) = \sum_{i=0}^{k-1} \binom{n}{i} p^i q^{n-i}.$$  

To find the expression for the sensitivity of system reliability on component reliability in this i.i.d. case, we can take the first derivative of $R(k, n)$ with respect to $p$. Using equation, we have

$$\frac{dR(k, n)}{dp} = k \binom{n}{k} p^{k-1} q^{n-k}. $$
Design of K out of n systems

- Optimal design for k & n to max system reliability, cost rate, optimal replacement times,... subject to restrictions on the system.
- Unconstrained optimization.
- Constrained optimization LP, NLP, GP,...
Optimal System Size $n$

In a $k$-out-of-$n$:G system, at least $k$ components need to work for the system to work. For a fixed $k$ value, the higher the system size $n$, the higher the reliability of the system. The difference between $n$ and $k$ represents the degree of redundancy built into the $k$-out-of-$n$:G system. However, as $n$ increases, there is a diminishing benefit for each additional component. In addition, the cost of the system increases as $n$ increases.

Model by Pham

For a $k$-out-of-$n$:G system with i.i.d. components, Pham proposes a model for determination of optimal system size $n$ and provides optimal solutions. The model assumptions are as follows:
1. All components are i.i.d. with reliability $p$ (unreliability $q$), $0 < p < 1$.
2. The cost of each component is $c$.
3. The cost of system failure is $d$.
4. A constant $k$ is given.
5. The objective is to find $n$ that minimizes the expected total cost, $E(T_n)$. 

The objective function to be minimized is

$$E(T_n) = cn + d(1 - R(k, n)) = cn + d \left(1 - \sum_{i=k}^{n} \binom{n}{i} p^i q^{n-i}\right).$$

There is only one decision variable, $n$, in this objective function. It may take only integer values in the range $[k, \infty)$. We will examine the trend of $E(T_n)$ as a function of $n$ by examining the increment in $E(T_n)$ when $n$ increases by one unit. Define
\[ \Delta E(T_n) \equiv E(T_{n+1}) - E(T_n), \]
\[ \Delta R_n(k, n) \equiv R(k, n + 1) - R(k, n). \]

Then, we have
\[ \Delta E(T_n) = c - d \Delta R_n(k, n). \]

Apparently, the trend of \( E(T_n) \) is heavily influenced by the trend of \( R(k, n) \). Based on equation, we have
\[ \Delta R_n(k, n) = R(k, n + 1) - R(k, n) = \binom{n}{k-1} p^k q^{n-k+1} \]
\[ = p \left( \frac{p}{q} \right)^{k-1} \binom{n}{k-1} q^n. \]
This quantity, $\Delta R_n(k, n)$, is itself a function of $n$. The increment in $\Delta R_n(k, n)$ can again be examined by allowing $n$ to increase by one unit to $n + 1$:

$$\Delta(\Delta R_n(k, n)) = \Delta R_n(k, n + 1) - \Delta R_n(k, n)$$

$$= p^k q^{n-k+1} \binom{n}{k-1} \left[ \frac{(n+1)q}{n-k+2} - 1 \right]$$

$$\geq 0 \quad \text{if and only if} \quad n \leq n_0,$$

where

$$n_0 = \left[ \frac{k-1}{p} - 1 \right].$$
In words, $\Delta R_n(k, n)$ is an increasing function of $n$ for $k \leq n \leq n_0$ and a decreasing function of $n$ for $n > n_0$.

To state the optimal solution, $n^*$, that minimizes the objective function given in, further define

$$n_a \equiv \inf \left\{ n \in [n_0, \infty) : \Delta R_n(k, n) < \frac{c}{d} \right\}.$$ 

In words, $n_a$ is the smallest integer $n$ such that $n \geq n_0$ and $\Delta R_n(k, n) < c/d$. The optimal integer solution, $n^*$, that minimizes the expected total system cost, $E(T_n)$, is given

Fix $p$, $k$, $d$, and $c$ with $\Delta R_n(k, n)$, $n_0$, and $n_a$ as defined in equations.
The optimal value $n^*$ such that the expected total cost of a $k$-out-of-$n$:G system is minimized is as follows:

1. If $\Delta R_n(k, n_0) < c/d$, then $n^* = k$.

2. If $\Delta R_n(k, n_0) \geq c/d$ and $\Delta R_n(k, k) \geq c/d$, then $n^* = n_0$.

3. If $\Delta R_n(k, n_0) \geq c/d$ and $\Delta R_n(k, k) < c/d$, then

$$n^* = \begin{cases} k & \text{if } E(T_k) \leq E(T_{n_0}), \\ n_0 & \text{if } E(T_k) > E(T_{n_0}). \end{cases}$$
Model by Nakagawa

Nakagawa assumes that the total cost of system failure is equal to \( nc_1 + c_2 \), which depends on system size. This cost includes replacement of failed components and inspection of nonfailed components. The mean cost rate is to be minimized in determination of system size \( n \).

Notation

- \( \lambda \): failure rate of each component
- \( c_1 \): acquisition cost of a component
- \( c_2 \): additional cost of the system which is replaced at failure

The mean cost rate is equal to the total cost of system failure divided by the MTTF of the system:

\[
C(n) = \frac{nc_1 + c_2}{(1/\lambda) \sum_{i=k}^{n} (1/i)}.
\]

Function \( C(n) \) is discrete. To find the optimal number \( n^* \) that minimizes \( C(n) \) for a given \( k \) value, we attempt to find the \( n \) value such that

\[
C(n + 1) \geq C(n),
\]

which is equivalent to

\[
L(n) \geq \frac{c_2}{c_1},
\]
where \( L(n) \) is defined as

\[
L(n) \equiv (n + 1) \sum_{i=k}^{n} \frac{1}{i} - n \quad \text{for } n \geq k.
\]

This newly defined function, \( L(n) \), is a monotonically increasing function over the definition domain of \( n \) because of the following observations:

\[
L(n + 1) - L(n) = \sum_{i=k}^{n+1} \frac{1}{i} > 0,
\]

\[
L(n) \geq \frac{n + 1}{k} - k \to \infty \quad \text{as } n \to \infty.
\]

As a result, there is a unique smallest \( n \) value that satisfies the condition in

Consequently, there is a unique smallest \( n \) value that satisfied the condition in

The optimal cost rate must be in the following range:

\[
\lambda c_1 n^* < C(n^*) \leq \lambda c_1 (n^* + 1).
\]
This can be easily verified by observing the following:

\[
\frac{C(n)}{\lambda c_1} = \frac{n + c_2/c_1}{\sum_{i=k}(1/i)} = \frac{n + c_2/c_1}{[n + L(n)]/(n + 1)} = (n + 1)\frac{n + c_2/c_1}{n + L(n)}.
\]

Substituting \( n^* \) into equation and noting that condition is satisfied, we have

\[
\frac{C(n^*)}{\lambda c_1} \leq n^* + 1.
\]

The other half of the inequality given in can be verified with the following:

\[
\frac{C(n^*)}{\lambda c_1} > \frac{C(n^* - 1)}{\lambda c_1} = n^* \times \frac{n^* - 1 + c_2/c_1}{n^* - 1 + L(n^* - 1)} > n^*.
\]

This model by Nakagawa has been extended to include two different types of failures by Sheu and Kuo. Whenever a component fails, there is a probability \( 1 - a \) (\( 0 < a < 1 \)) that it is a type A failure and a probability \( a \) that it is a type B failure. A type A failure is minimally repaired instantaneously with a cost \( c \) while a type B failure is left alone. The system is failed when the total number of failed components (due to type B failures) reaches \( n - k + 1 \). When the system fails, the total replacement cost is \( nc_1 + c_2 \). Under these model assumptions, the mean cost rate of the system is

\[
C(n) = \frac{\text{expected cost in a renewal cycle}}{\text{MTTF}} = \frac{nc_1 + c_2 + c(1 - a)(n - k + 1)/a}{(1/\lambda a) \sum_{i=k}(1/i)}.
\]
Define the function

\[ L(n) = \left[ c_1 + \frac{c(1 - a)}{a} \right] \left[ (n + 1) \sum_{i=k}^{n} \frac{1}{i} - n \right] + \frac{c(1 - a)(k - 1)}{a} \quad \text{for } n \geq k. \]

Function \( L(n) \) can be verified to be a strictly increasing function to infinity. To find the value of \( n \) such that \( C(n + 1) \geq C(n) \) and \( C(n - 1) < C(n) \) is equivalent to finding the value of \( n \) such that \( L(n) \geq c_2 \) and \( L(n - 1) < c_2 \). The optimal value \( n^* \) is equal to either \( k \) when \( L(k) > c_2 \) or the smallest \( n \) value that makes \( L(n) \geq c_2 \).
Optimal Replacement Time

Nakagawa considers the problem of determining optimal replacement time of a $k$-out-of-$n$:G system. The model assumptions are as follows:

- All components are i.i.d. following the exponential lifetime distribution with parameter $\lambda$.
- The system is replaced at time $T$ or at failure, whichever occurs first.
- The cost of replacing a nonfailed system is $nc_1$ while the cost of replacing a failed system is $nc_1 + c_2$.

The mean time to replacement of the system has an expression similar to the one given in equation. Thus, we can express the mean cost rate of the system as

$$C(T) = \frac{nc_1 + c_2(1 - R_s(T))}{\int_0^T R_s(t) \, dt},$$

where

$$R_s(t) = \sum_{i=k}^{n} \binom{n}{i} e^{-\lambda_i t} \left(1 - e^{-\lambda t}\right)^{n-i}.$$
Because a $k$-out-of-$n$:G system with i.i.d. exponential components has an increasing failure rate, there exists a finite optimal preventive replacement time. Using usual calculus techniques, if there exists an interior solution ($0 < T < \infty$) to maximize $C(T)$, it would be given by the equation

$$g(T) = \frac{nc_1}{c_2},$$

where

$$g(T) \equiv \frac{n\left(\frac{n-1}{k-1}\right) \sum_{i=k}^{n} \frac{1}{i} \binom{n}{i} \binom{i-1}{k-1} (-1)^{i-k} \left(1 - e^{-\lambda Ti}\right)}{\sum_{i=k}^{n} \binom{n}{i} \left(\frac{e^{-\lambda Ti}}{1-e^{-\lambda Ti}}\right)^{i-k}}$$

$$- \sum_{i=0}^{k-1} \binom{n}{i} e^{-\lambda Ti} \left(1 - e^{-\lambda T}\right)^{n-i}.$$
Function $g(T)$ has the following properties:

1. $g(0) = 0$,
2. $g(\infty) = \lim_{T \to \infty} g(T) = k \sum_{i=k}^{n} \left( \frac{1}{i} \right) - 1$, and
3. $g'(T) \geq 0$.

In words, $g(T)$ is a monotonically increasing function of $T$ approaching a constant value as $T$ goes to infinity. If $g(\infty) > n c_1 / c_2$, there exists an interior solution, $T^*$. Otherwise, $T^* \to \infty$. Define

$$
\psi = \left( \sum_{i=k+1}^{n} \frac{1}{i} \right) - \frac{nc_1}{kc_2}.
$$
If $\Psi > 0$, there exists a finite and unique $T^*$ that satisfies equation and minimizes $C(T)$. If $\Psi \leq 0$, then $T^* \to \infty$, namely, we should make no replacement of the system before failure. If $k = n$, the system is a series system that has an exponential distribution. Thus, no preventive replacement will be made either.

This work by Nakagawa has been extended by Sheu and Kuo to include two types of failure. In this case the objective function is

$$g(T) = \left[ nc_1 + c_2 \sum_{j=0}^{k-1} \binom{n}{j} e^{-a\lambda T} \left(1 - e^{-a\lambda T}\right)^{n-j} \right]$$

$$+ \frac{c(1-a)}{a} \sum_{j=0}^{n-1} \sum_{i=0}^{n-j-1} \binom{n}{i} e^{-a\lambda Ti} \left(1 - e^{-a\lambda T}\right)^{n-i} \right]$$

$$\times \left\{ \frac{1}{a\lambda} \left[ \sum_{j=k}^{n} \sum_{i=0}^{j-1} \binom{n}{i} e^{-a\lambda Ti} \left(1 - e^{-a\lambda T}\right)^{n-i} \right] \right\}^{-1}.$$

Standard calculus techniques are needed to find $T^*$ that minimizes $g(T)$. Sheu and Kuo also extend this model to the case when minimal repair costs are time dependent and stochastic.
Thanks for your time