Groups of small Morley rank

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Abstract

Notes for a mini-course given at Universidad de los Andes in October 2018. There were five lectures of two hours each, devoted to proving the Cherlin-Zilber conjecture in rank 3.

Contents

Introduction

Groups of finite Morley rank are the model-theorist's approach to algebraic groups over algebraically closed fields: one has a notion of dimension (the Morley rank) which behaves like the Zariski dimension, but there is no given topological/functorial structure; this allows only basic arguments.

The Cherlin-Zilber Conjecture is however that all infinite simple groups of finite Morley rank ought to be algebraic groups over algebraically closed fields. Despite nearly 40 years of efforts, the conjecture is still open. As a matter of fact, the solution in rank 3 was given only two years ago by Frécon. It is completely independent from the bulk of earlier work (mostly finite group-oriented), and it is unclear whether it will open new paths. But Wagner's rewriting of Frécon's theorem is a beautiful and significant piece of mathematics, which can be taught in a self-contained class.

The course will thus explain the full solution of the conjecture in rank 3. It is of interest to at least four (non-disjoint) kinds of mathematicians:

- model-theorists: who will understand what is going on since no big group-theoretic guns are required in small rank;
- group-theorists: who will enjoy seeing how important involutions are even in mathematical logic;
- geometers: who will be puzzled by what happens when one looks at algebraic groups from the model-theoretic perspective;
- aesthetes: since, needless to say, the methods are beautiful.

These notes should be fairly self-contained; I have also tried to provide historical comments, research directions, and exercises.

Lecture 1 – The Cherlin-Zilber Conjecture

In this lecture. The first lecture contains no technical, but cultural material. We follow the algebraic approach to groups of finite Morley rank: there are only three simple, self-contained definitions. Then we discuss the Cherlin-Zilber conjecture with some variations, and the state of the art.

References:
- An extremely well-explained reference is [**Borovik-Nesin**, §4].
- More details on model theory in [**Poizat**, Introduction].

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1 Model-theoretic background

There will be little model theory in this class as we are mostly concerned with group-theoretic aspects. The dimension discovered by Michael Morley for the purpose of pure logic, turned out to be deeply algebraic in nature; we shall please all parties by saying that the study of groups of finite Morley rank belongs to *model-theoretic algebra*.

We follow the axiomatic approach suggested by Borovik and completed by Poizat. It is followed in Borovik's and Nesin's excellent book [Borovik-Nesin]. For orthodox model theorists, the proof of the legitimacy of this approach is fairly involved, and can be found in Poizat's monograph [Poizat]. (As for algebraic groups, we shall make only moderate use of Humphrey's elegant and very accessible text [**Humphreys**].)

1.1 Definable and interpretable sets

Definition (structure)**.** *A* structure *on a set endows it with relations, all finitary, and among which a (unary, "Dirac-style") relation for each singleton.*

A group structure *is a structure on a group, having at least* $=$ *(which is a binary relation) and* $\cdot$ *(which is a ternary relation).*

Example. A special case of a group structure is a field $(\mathbb{K}, +, \cdot)$: one actually considers the additive group $\mathbb{K}_+ = (\mathbb{K}, +)$, with $\cdot$ as an extra (ternary) relation.

Definition (definable universe)**.** *The* definable universe *of a structure, with members called the* definable sets*, is the smallest collection* $\mathcal{U}$ *of sets:*

- *containing all relations (in particular, singletons are definable as one checks by returning to the earlier definition);*
- *closed under (finitary) algebraic combinations (i.e. boolean operations);*
- *closed under (finitary) Cartesian operations (i.e. taking finite Cartesian products but then also projections);*
- *closed under quotients (i.e. if* $A, E \in \mathcal{U}$ *and* $E \subseteq A^2$ *is an equivalence relation on* A *then* $A/E \in \mathcal{U}$*).*

As always in model-theoretic algebra, "definable" is what the orthodox model-theorist will call "intepretable with parameters". Our abuse is common, and systematic in sources such as [Borovik-Nesin]; for algebraists and geometers, it is rather pointless to distinguish levels. One may wish that some day, model-theoretic terminology might evolve towards something less rebuking.

[Borovik-Nesin]: Alexandre Borovik and Ali Nesin. *Groups of finite Morley rank.* Vol. 26. Oxford Logic Guides. The Clarendon Press – Oxford University Press, New York, 1994. xviii+409 pages
[Poizat]: Bruno Poizat. *Stable groups.* Vol. 87. Mathematical Surveys and Monographs. Translated from the 1987 French original by Moses Gabriel Klein. American Mathematical Society, Providence, RI, 2001. xiv+129 pages
[Humphreys]: James Humphreys. *Linear algebraic groups.* Vol. 21. Graduate Texts in Mathematics. Springer-Verlag, New York, 1975. xiv+247 pages

Example (for geometers)**.** Consider an *algebraically closed* field $\mathbb{K}$. Then the definable universe of $\mathbb{K}$ is exactly the constructible class. Logicians call this elimination of quantifiers and of imaginaries: indeed, closedness under projections is Chevalley's elimination theorem, whereas closedness under quotients is not entirely trivial and essentially due to Weil.

For an arbitrary field the definable universe can be arbitrary complicated: it so happens that the integers are definable in $(\mathbb{Q}, +, \cdot)$.

If you come from group theory, just bear in mind that *not all* subsets of a given group are definable. For instance, given a group G, the commutator subgroup G' need not be definable. Likewise, a Sylow 2-subgroup need not be definable. But provided $H < G$ is a definable subgroup, the quotient set G/H is definable as well.

1.2 The Rank function

Definition (rank function; Borovik-Poizat axioms)**.** *Let $\mathcal{M}$ be a structure and $\mathcal{U}$ be its definable universe. A* rank function *is a finite-valued map* $\text{rk} : \mathcal{U} \setminus \{\emptyset\} \to \omega$ *with the following properties, in which A, B stand for definable sets and $f : A \twoheadrightarrow B$ is a definable surjection:*

- $\text{rk}\, A \geq n+1$ *iff there are infinitely many disjoint definable $B_i \subseteq A$ with* $\text{rk}\, B_i \geq n$ *("monotonocity", but shold be called "inductive definition");*
- *for every integer k, the set* $F_k := \{b \in B : \text{rk}\, f^{-1}(b) = k\}$ *is definable ("definability");*
- *if $B = F_k$ for some k then* $\text{rk}\, A = k + \text{rk}\, B$ *("additivity");*
- *there is an integer ℓ such that for each $b \in B$, either $f^{-1}(b)$ has at most ℓ elements or is infinite ("elimination of infinity").*

If G is a group whose universe bears a rank function, we call G a *ranked group* or a *group of finite Morley rank*. Despite the name, it is to be thought of as a dimension.

Examples.

- Any group definable in the universe of an $\aleph_1$-categorical structure is a group of finite Morley rank. This is how the subject started: Zilber was essentially trying to understand uncountably categorical structures, after the seminal, abstract work by Morley.

  This example need not be understood as I shall not define categoricity.
- Any group definable in a ranked universe, is itself a group of finite Morley rank (with respect to the inherited definable structure, and the same rank function).
- Any algebraically closed field is a group of finite Morley rank: with rank function the Zariski dimension (on the constructible class).
- By the last two, one gets all algebraic groups as examples. More specifically:

  Let $\mathbb{K}$ be an algebraically closed field, $\mathbb{G}$ be an algebraic group ("defined over $\mathbb{K}$") and $G = \mathbb{G}(\mathbb{K})$ be the group of $\mathbb{K}$-points of $\mathbb{G}$. Then G is a group of finite Morley rank where the definable universe is the constructible class, and the rank function the Zariski dimension.

  To logicians the algebraic, functorial terminology and notation may look frightening but they are not. For instance, PGL_2 is a simple algebraic group; $\text{PGL}_2(\mathbb{C})$ may be familiar (an algebraic group is a functor, say a recipe for producing a group once a coefficient ring is given).

2 A Discussion of the conjecture

2.1 Statement and variations

Given the above modern exposition, it is a trivial matter to conjecture that groups of finite Morley rank have a lot to do with algebraic geometry. But things were different in the seventies; as a matter of fact it took a couple of years to realise that Morley rank was a generalisation of the Zariski dimension in algebraically closed fields, so it took a while to figure out that groups of points $\mathbb{G}(\mathbb{K})$ had finite Morley rank indeed. But Zilber had the insight that $\aleph_1$-categoricity *is* algebraic geometry; a deep idea which we now know to be slightly wrong, but which triggered fascinating research. On the other hand Cherlin, although he could not understand all groups of rank 3, decided there ought to be a reasonable classification.

Conjecture (Cherlin [**Che79**], Zilber [**Zil77**])**.** *An infinite simple group of finite Morley rank is, as a group, the group of* $\mathbb{K}$*-points of some algebraic group* $\mathbb{G}$ *in some algebraically closed field* $\mathbb{K}$*.*

Immediate note: without simplicity it is trivially false, as demonstrated by $\mathbb{Q}_+$ for instance, or $\mathbb{G}(\mathbb{C}) \times \mathbb{G}(\overline{\mathbb{F}_2})$. The purpose of this class is to prove the conjecture in rank 3; in higher rank it is open. Before moving to the strategy let me state a few analogue statements, some open, some not.

Theorem (Macintyre [**Mac71**]; Cherlin-Shelah [**CS80**])**.** *An infinite skew-field of finite Morley rank is an algebraically closed (commutative) field.*

Macintyre's theorem is older than the conjecture by a couple of years. After Artin and Wedderburn this quickly yields the following.

Theorem (Zilber [**Zil74**, Theorem 2]; Felgner [**Fel75**, Satz 3])**.** *An infinite simple (associative, non-commutative) ring of finite Morley rank is a full matrix ring over an algebraically closed field.*

Closer to groups are however Lie rings, which are not associative. A *Lie ring* is a $(+, [\cdot,\cdot])$-structure where $+$ defines an abelian group while $[\cdot,\cdot]$ is bi-additive, skew-symmetric and satisfies the Jacobi identity. Any genuine Lie algebra reduces to a Lie ring; however the linear structure is not encoded inside the Lie ring.

Conjecture. *An infinite simple Lie ring of finite Morley rank is, as a Lie ring, the Lie ring of* $\mathbb{K}$*-points of some simple Lie algebra* $\mathfrak{g}$ *in some algebraically closed field* $\mathbb{K}$*.*

To my knowledge, this conjecture has received almost no attention except for the PhD of Rosengarten [**Ros91**], providing a positive answer in rank ≤ 3. We shall return to it in the final notes.

Let us move to other contexts.

Theorem (Peterzil, Pillay, Starchenko [**PPS00**, Corollary 4.4])**.** *A simple group definable in an o-minimal theory is, as a group, the group of* $\mathcal{R}$*-points of some semi-algebraic group* $\mathbb{H}$ *in some real closed field* $\mathcal{R}$*.*

Proving the above strongly requires the order structure: with the ordering, once one gets a field one also has infinitesimals and calculus, hence differential geometric techniques such as the Lie algebra. *This is not the case in the finite Morley rank context*, which is one of the reasons the Cherlin-Zilber conjecture still stands open.

It should be a good exercise to prove the following, presumably a subset of existing work.

Fact (for which I know no reference)**.** *An infinite simple Lie ring definable in an o-minimal theory is a Lie algebra over some real closed closed field.*

Be careful that the above does *not* say that the Lie ring is the ring of $\mathcal{R}$-points of some simple Lie algebra: we would be missing $\mathfrak{so}_3$.

Past this we enter the realm of fancy.

Conjecture. *There are natural model-theoretic assumptions under which a p-adic analogue of the Cherlin-Zilber conjecture is both interesting and not trivially false.*

[**Che79**]: Gregory Cherlin. 'Groups of small Morley rank'. *Ann. Math. Logic* 17(1-2) (1979), pp. 1–28

[**Zil77**]: Boris Iossifovitch Zilber. 'Groups and rings whose theory is categorical'. *Fund. Math.* 95(3) (1977), pp. 173–188

[**Mac71**]: Angus Macintyre. 'On ω_1-categorical theories of fields'. *Fund. Math.* 71(1) (1971), 1–25. (errata insert)

[**CS80**]: Gregory Cherlin and Saharon Shelah. 'Superstable fields and groups'. *Ann. Math. Logic* 18(3) (1980), pp. 227–270

[**Zil74**]: Boris Iossifovitch Zilber. 'Rings whose theory is $\aleph_1$-categorical'. *Algebra i Logika* 13 (1974), pp. 168–187, 235

[**Fel75**]: Ulrich Felgner. '$\aleph_1$-Kategorische Theorien nicht-kommutativer Ringe'. *Fund. Math.* 82 (1974/75). Collection of articles dedicated to Andrzej Mostowski on the occasion of his sixtieth birthday, VIII, pp. 331–346

[**Ros91**]: Richard Rosengarten. '$\aleph_0$-stable Lie algebras'. PhD thesis. New Brunswick: Rutgers, The State University of New Jersey, 1991. 62 pp.

[**PPS00**]: Yaacov Peterzil, Anand Pillay and Sergei Starchenko. 'Definably simple groups in o-minimal structures'. *Trans. Amer. Math. Soc.* 352(10) (2000), pp. 4397–4419

2.2 Failed attempts

I will briefly discuss how *not* to prove the Cherlin-Zilber conjecture. It does not mean that the ideas in this vein are silly; there are just too many technical obstructions.

Character theory is not first-order. Lie theory is based either analytically on infinitesimals and calculus, which is essentially incompatible with stability, or algebraically on functoriality and the ability to consider $\mathbb{G}(\mathbb{K}[\varepsilon])$, which is not an elementary extension. So unless one comes with an outstanding method bridging the gap, both theories are unaccessible in the study of groups of finite Morley rank.

Let me mention three attempts.

Naive way: induction. Nothing great was ever proved by induction. As a matter of fact the solution in rank 3 (the purpose of this class) came as late as 2016. Rank 4 is open.

Geometric way: linearisation techniques. The first studies by Zilber may suggest a tentative roadmap.

1. Find an algebraically closed field $\mathbb{K}$ inside G.

   Problem: it is open whether every infinite simple group of finite Morley rank defines an infinite field.

   The most celebrated result in this direction, though by no means the only one, is Zilber's "Field Theorem" [**Zil84**, Lemma 2], [Borovik-Nesin, Theorem 9.1] (in effect an astonishing combination of the Chevalley-Zilber generation lemma known as "Zilber's indecomposibility theorem" [Humphreys, Proposition 7.5], [Zil77, Theorem 3.3], or [Borovik-Nesin, §5.4], and the Schur-Zilber lemma on definability of covariance skew-fields), stating that every infinite connected soluble, non-nilpotent group of finite Morley rank does. Other results exist but experts have historically been obsessed with this one.

   At the non-simple level, this is simply not true: Baudisch [**Bau96**] could construct nilpotent, non-abelian groups of finite Morley rank which do *not* define an infinite field. (It it well-known to geometers, though unfortunately not explicit in [Humphreys], that these can therefore *not* be algebraic group, but bear in mind that the Cherlin-Zilber conjecture is stated for simple groups.)

   Still: let us assume that G does define an infinite field $\mathbb{K}$; by Macintyre's theorem on fields [Borovik-Nesin, Theorem 8.1], $\mathbb{K}$ is algebraically closed.

2. Define G inside $\mathbb{K}$.

   One can do this using either Zilber's indecomposibility theorem, or an alternative approach due to Hrushovski [Poizat, §2.6].

   Problem: G will be definable in ($\mathbb{K}$; G-induced structure), but it has no reason to be definable in the mere field structure $(\mathbb{K}; +, \cdot)$. Because of work by Hrushovski and followers on fields [**Hru92**], we know that the definable universe of a field of finite Morley rank can be fairly complicated—in particular that $\mathbb{K}_+$ and $\mathbb{K}^\times$ need not be 1-dimensional groups, as opposed to algebraic geometry where they are the basic building blocks [Humphreys, Theorem 20.5].

   So we do not know what we gain exactly from defining G in a field structure.

3. Make G definably linear (i.e. definable in $\mathrm{GL}_n(\mathbb{K})$).

   Problem: there is no way to linearise (even dropping definability) abstract groups of finite Morley rank since we have no form of Lie theory.

   Still: assuming $G \leq \mathrm{GL}_n(\mathbb{K})$ definably, one can use linear algebra. As a matter of fact, Poizat [**Poi01**] proved that in positive characteristic, we would be done, i.e. G must be algebraic. This is not known if $\mathbb{K}$ has characteristic 0 but we still have strong constraints on the structure of G.

[Zil84]: Boris Iossifovitch Zilber. 'Some model theory of simple algebraic groups over algebraically closed fields'. *Colloq. Math.* 48(2) (1984), pp. 173–180

[Bau96]: Andreas Baudisch. 'A new uncountably categorical group'. *Trans. Amer. Math. Soc.* 348(10) (1996), pp. 3889–3940

[Hru92]: Ehud Hrushovski. 'Strongly minimal expansions of algebraically closed fields'. *Israel J. Math.* 79(2-3) (1992), pp. 129–151

[Poi01]: Bruno Poizat. 'Quelques modestes remarques à propos d'une conséquence inattendue d'un résultat surprenant de Monsieur Frank Olaf Wagner'. *J. Symbolic Logic* 66(4) (2001), pp. 1637–1646

Despite work by Altınel and Wilson [**AW09**],[**AW11**], perhaps linearisation did not receive sufficient attention yet. But there are two many obstacles in this direction.

Model-theoretic way: locally finite transfer. The idea is simple: "transfer down" from an arbitrary group of finite Morley rank to a *locally finite* model. Since simplicity of a group is preserved by elementary extensions (a non-trivial result by Zilber), we would then be able to use the following consequence of the classification of the finite simple groups (CFSG).

Fact (S. Thomas [**Tho83**]; also Belyaev [**Bel84**] and Hartley-Shute [**HS84**] – *all use* CFSG)**.** *An infinite, locally finite, simple group with the chain condition on centralisers is the group of* $\mathbb{K}$*-points of a (possibly twisted) Chevalley group in a locally closed field* $\mathbb{K}$*.*

Corollary. *An infinite* locally finite *simple group of finite Morley rank is, as a group, the group of* $\mathbb{K}$*-points of some algebraic group* $\mathbb{G}$ *in some algebraically closed field* $\mathbb{K}$*.*

Quite interestingly, Borovik could prove the following without invoking the classification of the finite simple groups, only its methods.

Fact (Borovik [**Bor95**, Theorem 7.1] – does *not* use CFSG)**.** *An infinite locally finite simple group of finite Morley rank not containing* $\oplus_{\mathbb{N}}\mathbb{Z}/2\mathbb{Z}$ *is, as a group, the group of* $\mathbb{K}$*-points of some algebraic group* $\mathbb{G}$ *in some algebraically closed field* $\mathbb{K}$*.*

Unfortunately transfer is simply impossible, except (as of today) in two special cases:

- Definability of the group inside a field: this is how using a result on the model theory of fields by Wagner [**Wag01**], Poizat could prove his linear theorem [Poi01].
- Presence of special, non-definable automorphisms which have a nice group of fixed points. This approach was recently revived by Karhumäki [**Kar18**].

In general this seems hopeless. But it is interesting how the finite connection and the classification of the finite simple groups naturally appeared in a model-theoretic topic.

2.3 The Borovik programme

The finite analogue. Finite group theory culminates in the following classification result.

Fact ("CFSG": Brauer, Galois, Gorenstein, Jordan, Mathieu, . . .)**.** *Let* G *be a finite simple group. Then one of the following holds:*

- $G \simeq \mathbb{Z}/p\mathbb{Z}$ *for prime* p*;*
- $G \simeq \mathrm{Alt}_n$ *for* $n \geq 5$*;*
- G *is a group "of Lie type" i.e. a (possibly twisted) Chevalley over a finite field;*
- G *is one of the 26 so-called "sporadic" groups.*

The project was started by Brauer and announced fulfilled by Gorenstein in the 1980's; it ranges over 10,000 pages. To understand the analogy with our setting notice that:

- there are no simple infinite *abelian* groups;
- $\mathrm{Alt}_{\mathbb{N}}$ is not a group of finite Morley rank (it is model-theoretically ugly: it has IP);
- for technical reasons, and without any definition, there are no definable "twists" of fields of finite Morley rank;

[AW09]: Tuna Altınel and John Wilson. 'On the linearity of torsion-free nilpotent groups of finite Morley rank'. *Proc. Amer. Math. Soc.* 137(5) (2009), pp. 1813–1821

[AW11]: Tuna Altınel and John Wilson. 'Linear representations of soluble groups of finite Morley rank'. *Proc. Amer. Math. Soc.* 139(8) (2011), pp. 2957–2972

[Tho83]: Simon Thomas. 'The classification of the simple periodic linear groups'. *Arch. Math. (Basel)* 41(2) (1983), pp. 103–116

[Bel84]: Vissarion Viktorovich Belyaev. 'Locally finite Chevalley groups'. In: *Studies in group theory.* Akad. Nauk SSSR, Ural. Nauchn. Tsentr, Sverdlovsk, 1984, pp. 39–50, 150

[HS84]: Bryan Hartley and Gary Shute. 'Monomorphisms and direct limits of finite groups of Lie type'. *Quart. J. Math. Oxford Ser. (2)* 35(137) (1984), pp. 49–71

[Bor95]: Alexandre Borovik. 'Simple locally finite groups of finite Morley rank and odd type'. In: *Finite and locally finite groups (Istanbul, 1994).* Vol. 471. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Kluwer Acad. Publ., Dordrecht, 1995, pp. 247–284

[Wag01]: Frank Wagner. 'Fields of finite Morley rank'. *J. Symbolic Logic* 66(2) (2001), pp. 703–706

[Kar18]: Ulla Karhumäki. 'A model theoretic approach to simple groups of finite Morley rank with finitary groups of automorphisms'. Preprint. arXiv 1801.00576 (Modnet 1359). 2018

- at this stage, or perhaps due to our lack of imagination, one can hope that infiniteness will simply smooth out the sporadics.

Borovik had the idea to systematically borrow methods from the (then recently solved) CFSG to attack the (then recently posed) algebraicity conjecture. This is called the "Borovik programme". Although the present course deals with groups too small to implement it, this orthodox approach is at the core of the topic and I wish to say a few cultural words.

Involutions and strongly real elements. It is hardly exaggerated to claim that all the classification of the finite simple groups builds on Sylow theory, and the distribution of *involutions* and *strongly real elements.*

Definition (involution)**.** *An* involution *of a group is an element of order* 2*.*

Fun fact: the word was coined by Girard Desargues.

Definition (strongly real element)**.** *A* strongly real element *is a product of two distinct involutions.*

Fundamental fact: let i and j be involutions. Then:

$$(ij)^i = i \cdot ij \cdot i = j \cdot i = (ij)^{-1},$$

viz. *two involutions invert their product.*

Sylow theory. It is not uninteresting to note that the Sylow theorems in finite group theory were proved the same year as the Erlangen programme was written, viz. 1872.

Definition (Sylow 2-subgroup)**.** *A Sylow* 2*-subgroup of a group G is a* 2*-subgroup maximal wrt inclusion.*

This always exists. Now if G is a *finite* group, then Sylow 2-subgroups are:

- definable (they are finite);
- nilpotent (they are finite p-groups);
- conjugate;
- of order a function of the order of G (but not of its structure).

Now at the infinite level, if G is a group of finite Morley rank:

- Sylow 2-subgroups need not be definable;
- Sylow 2-subgroups are nilpotent (not known for $p > 2$);
- Sylow 2-subgroups are conjugate [**BP90**] (not known for $p > 2$);
- Sylow 2-subgroups could be trivial even if G is simple.

The first point confirms that since one must leave the definable category, the topic of groups of finite Morley rank does not pertain to "pure" model-theoretic algebra. The last constrats with finite group theory.

Fact (Feit-Thompson "odd order theorem")**.** *Let G be a non-abelian, finite simple group. Then G has an involution.*

Also quoted as: "a finite group of odd order is soluble". The proof is a mild 250 pages and uses character theory. *There is no analogue in the theory of groups of finite Morley rank.*

Victories and front lines of the Borovik programme. Since we have no Feit-Thompson theorem for groups of finite Morley rank, one must *assume* that G has involutions. Even then the situation is fairly different depending on the structure of the Sylow 2-subgroup (singular is legitimate by conjugacy).

Theorem (Altınel-Borovik-Cherlin [**ABC08**])**.** *Let G be a simple group of finite Morley rank. Suppose that G contains a copy of $\oplus_{\mathbb{N}}\mathbb{Z}/2\mathbb{Z}$. Then $G \simeq \mathbb{G}(\mathbb{K})$ for an algebraic group $\mathbb{G}$ and an algebraically closed field $\mathbb{K}$* of characteristic 2*.*

[BP90]: Aleksandr Vasilievich Borovik and Bruno Petrovich Poizat. 'Tores et p-groupes'. *J. Symbolic Logic* 55(2) (1990), pp. 478–491
[ABC08]: Tuna Altınel, Alexandre Borovik and Gregory Cherlin. *Simple groups of finite Morley rank.* Vol. 145. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 2008. xx+556 pages

This covers a whole case of the Cherlin-Zilber conjecture and is only 500 pages long, which is short with respect to finite group standards. Interestingly enough, in characteristic 2, the Sylow 2-subgroups *are* definable.

The state of the art in the remaining case is as follows.

Theorem (Burdges [**Bur09**])**.** *Let G be a simple group of finite Morley rank. Suppose that G is a minimal counter-example to the Cherlin-Zilber conjecture, viz. every simple, definable subquotient is algebraic. Then the Sylow 2-subgroups are not too big (in a precise sense).*

But still, one knows almost nothing of groups with no involutions.

Horizons. Let me finish with two not-so-unreasonable perspectives:

- progress on the minimal configurations of Deloro-Jaligot [**DJ16**], in particular using the geometry of involutions. This will be discussed in the fourth lecture;
- the hope that the study of groups of small rank will yield general ideas and a machinery. For the moment it is not the case and the rank 3 argument does not generalise nor even suggest a strategy in rank 4. This will be seen in the last lecture. For the moment this path is strictly orthogonal to Borovik's, which in my opinion makes it dubious.

In any case we are still fighting and I will definitely *not* give the same class in ten years.

Final notes and exercises

The present section and its siblings contain material not discussed in class; as a matter of fact, the reader can freely omit them although the enthusiast may find relevant material here.

A few words on the history of the topic

It is the next generation's duty to become the mythographers of their greater predecessors.

It is said that everything started with Michael Morley [**Mor65**]. Morley was the Colombus of geometric model theory.

Zilber was its Vespucci. He apparently was the first to envision the deep relationship between algebraic geometry and model theory, as early as the 1970's. In the West another branch of model theory, viz. classification theory, was then under spotlight—which may be one of the reasons Zilber's work remained somehow confined in the Soviet Union. In any case Cherlin's first article on groups of finite Morley rank was written independently of Zilber's work.

Of the Novosibirsk school (just a few years after Zilber) were Belegradek and of course Borovik. I once heard that Borovik, a PhD student working on the CFSG when Gorenstein announced it was over, had to come up with another mathematical interest: and recreated the topic of groups of finite Morley rank, from an algebraist's perspective, with a gap in his axioms fixed by Poizat and discussed below.

Then things accelerated with Nesin's PhD under Macintyre on the topic, and with Poizat's legendary visit to Novosibirsk, as he was preceded by the French edition of his book.

Borovik rank and Morley rank

In general, model-theoretic discussions are to be found in [Poizat], not in [Borovik-Nesin].

- The contemporary approach to groups of finite Morley rank is due to Borovik's inspiration. In his book Poizat calls groups with a ranked universe *Borovik groups*; he bridged the gap between Borovik groups and groups of finite Morley rank. As a matter of fact Borovik had identified the first three axioms but, not being a logician, he was missing the last, which was added by Poizat. Equivalence of Borovik's point of view and the classical model-theoretic setting is highly non-trivial; *it is not true without the group law.*

  Theorem (Poizat; [Poizat, §2.4])**.** *Let G be a group structure. Then the theory of G has finite Morley rank iff the universe of G has a rank function; in which case* $\mathrm{MR} = \mathrm{rk}$ *everywhere.*

  This also applies to "enriched" group structures: for instance to field structures as well.
- The theorem is non-trivial for two reasons:
  - it is not clear whether in a theory of finite Morley rank, MR extends nicely to *interpretable* sets;

[Bur09]: Jeffrey Burdges. 'Signalizers and balance in groups of finite Morley rank'. *J. Algebra* 321(5) (2009), pp. 1383–1406

[DJ16]: Adrien Deloro and Éric Jaligot. 'Involutive automorphisms of $N_\circ^\circ$-groups of finite Morley rank'. *Pacific J. Math.* 285(1) (2016), pp. 111–184

[Mor65]: Michael Morley. 'Categoricity in power'. *Trans. Amer. Math. Soc.* 114 (1965), pp. 514–538

- it is not even clear in the first place whether a Borovik group is a group of finite Morley rank, as the rank axioms do *not* require ω-saturation—it is not clear why is bearing a rank function a "strong" property, i.e. a property of the theory and not of the mere model.

The proof involves a thorough analysis of groups with a suitable dimension on definable sets (something common to both groups of finite Morley rank and Borovik groups, without yet knowing equivalence). We can but direct to Poizat's book [Poizat, §2] for details.
As a corollary to Poizat's theorem, both points of view (orthodox model-theoretic, revisionist algebraic) can be adopted in the study of groups of finite Morley rank/groups with a rank function. And since MR must then be the only rank function, it is safe to say "rank" with no reference to Michael Morley. It is historically fair to call the axioms the *Borovik-Poizat axioms.* But it still is quite misleading to call rank a dimension.

- Some day, experts will have to reflect upon the pros and contras (in very commercial terms) of advertising the domain under a name which directly refers to hard-core model theory, while none is really needed. Of course "groups with a dimension" is not precise enough to describe the specific behaviour of the Zariski dimension and the inductive definition. Tentative names: (finite) Morley rank is an *inductive dimension*, or an *algebraic dimension.*

Borovik rank and stability

- It is well-known that a structure of finite Morley rank must be stable, and even ω-stable. The careful model theorist will understand the following subtleties: since having a Borovik rank does not necessarily carry to elementary extensions, a non-saturated structure having a Borovik rank need not be of finite Morley rank, so little is known of the theory itself. However, Burdges-Cherlin [**BC02**] proved that a structure with a Borovik rank function must be stable, and even superstable. Interestingly ω-stability may fail.
- Some thirty years ago it was certainly illuminating to think of algebraically closed fields as the paradigm of stable theories. We now know that stable nature is fairly more complicated; any stable version of the Cherlin-Zilber is most likely to be false, because of the following striking result.

  Theorem (Sela [**Sel13**]; also work by Kharlampovitch-Myasnovik [**KM06**])**.** *The free group on $n > 2$ generator is stable.*

  I am certainly *not* suggesting that the free group is simple; but as one sees, stability goes far beyond the quietness of algebraically closed fields.

More on Sylow theory

- Sylow theory for $p = 2$ is very satisfactory, except for the desirable non-triviality of Sylow 2-subgroups, à la Feit-Thompson. All we know is the following. Connectedness is defined in the next lecture.

  Theorem (Borovik-Burdges-Cherlin [**BBC07**])**.** *Let G be a connected group of finite Morley rank. If Sylow 2-subgroups are finite, then they are trivial.*
- In general the situation is astoundingly messy. Conjugacy is open, except in the following cases:
  - if $p = 2$ [Borovik-Nesin, Theorem 10.11], first obtained by Borovik-Poizat [BP90];
  - if G is soluble [Borovik-Nesin, Theorem 9.35], a result of Altınel-Cherlin-Corredor-Nesin [**ABCC03**];
  - if G contains no infinite p-subgroup of finite exponent, proved by Burdges-Cherlin [**BC09**].

  Even worse: a finite p-group is always nilpotent, but this is not true of an infinite group, as demonstrated by the "Tarski monsters" first constructed by Olshanski [**Ols79**]. However a p-subgroup of an algebraic group is nilpotent. *We do not know whether a Sylow p-subgroup of a group of finite Morley rank must be nilpotent.* [Borovik-Nesin, §6.4] discusses the problem.

[BC02]: Jeffrey Burdges and Gregory Cherlin. 'Borovik-Poizat rank and stability'. *J. Symbolic Logic* 67(4) (2002), pp. 1570–1578
[Sel13]: Zlil Sela. 'Diophantine geometry over groups VIII: Stability'. *Ann. of Math. (2)* 177(3) (2013), pp. 787–868
[KM06]: Olga Kharlampovich and Alexei Myasnikov. 'Elementary theory of free non-abelian groups'. *J. Algebra* 302(2) (2006), pp. 451–552
[BBC07]: Alexandre Borovik, Jeffrey Burdges and Gregory Cherlin. 'Involutions in groups of finite Morley rank of degenerate type'. *Selecta Math. (N.S.)* 13(1) (2007), pp. 1–22
[ABCC03]: Tuna Altınel et al. 'Parabolic 2-local subgroups in groups of finite Morley rank of even type'. *J. Algebra* 269(1) (2003), pp. 250–262
[BC09]: Jeffrey Burdges and Gregory Cherlin. 'Semisimple torsion in groups of finite Morley rank'. *J. Math Logic* 9(2) (2009), pp. 183–200
[Ols79]: Alexander Olshanski. 'Infinite groups with cyclic subgroups'. *Dokl. Akad. Nauk SSSR* 245(4) (1979), pp. 785–787

$\aleph_0$- and $\aleph_1$-categorical structures

Macintyre [Mac71] proved algebraic closedness of a commutative field and asked whether commutativity always held. A positive answer was given by Zilber [Zil77], but also independently by Shelah [**She75**, Theorem 7.3] and Cherlin [**Che78**].

On more geenral structures, one will find in the introduction of a classical article by Baur-Cherlin-Macintyre [**BCM79**] a very useful summary, which has not much evolved since. [She75, §7] is also of interest. The topic of rings is neither trivial nor dead, as witnessed by recent work by Myasnikov and Sohrabi [**MS17**].

And beyond. . .

How much model theory is needed in order to classify the simple groups definable in *o*-minimal theories is not completely clarified: an algebraic rewriting could be a decent exercise. Other model-theoretic contexts are legitimate. At the moment of giving this class, it is unclear whether the notion of a *finite-dimensional theory* as introduced by Wagner [**Wag18a**] will prove robust enough to obtain conjugacy of the Sylow 2-subgroups. If not, then it may be regarded pointless by group theorists.

Exercise. All objects below are supposed to be definable.

- $\operatorname{rk} A = 0$ iff A is finite.
- If $B \subseteq A$ then $\operatorname{rk} B \leq \operatorname{rk} A$.
- $\operatorname{rk}(A \cup B) = \max(\operatorname{rk} A, \operatorname{rk} B)$.
- $\operatorname{rk}(A \times B) = \operatorname{rk} A + \operatorname{rk} B$.
- If $f : A \to B$, then $\operatorname{rk}(f(A)) \leq \operatorname{rk} A$; equality holds if f is injective but the converse need not be true.

Exercise. Consider the property (*):

for any two involutions i, j of a *finite* group, at least one of the following holds:

- i and j are conjugate;
- there is an involution k commuting with both i and j.

(i) Prove that if G is finite, then (*) holds.

(ii) Incidently, prove that in $\mathrm{SO}_3(\mathbb{R})$, *both* ads in (*) hold.

(iii) Suppose that G is a finite group with at least two non-conjugate involution. Prove that if m is the maximal order of the centraliser of an involution of G, then $|G| < m^3$. (Hint: consider the map $(i, j) \mapsto k$ where i and j range in distinct conjugacy classes of involutions, and k is an involution commuting to both.)

Actually (*) holds in groups of finite Morley rank. This requires the following fact: a group of finite Morley rank with no involutions is 2-divisible. We shall return to it in another exercise.

Exercise. Let G be a group of finite Morley rank and $i \in G$ be an involution. Show that there is a strongly real element $x \in i^G \cdot i^G$ such that:

$$\operatorname{rk}(G) \leq \operatorname{rk} C_G(x) + 2 \operatorname{rk} C_G(i)$$

Exercise. Let G be a group.

1. Prove that if G is finite then it *cannot* be covered by conjugates of a proper subgroup.
2. Prove that it can be the case if G is infinite.
3. (Harder.) Prove that infinite G can even be *partitioned* by conjugates of a proper subgroup.

The last property is open for G of finite Morley rank. This would provide a radical counterexample to Cherlin-Zilber.

[She75]: Saharon Shelah. ‘The lazy model-theoretician's guide to stability’. *Logique et Analyse (N.S.)* 18(71-72) (1975). Comptes Rendus de la Semaine d'Étude en Théorie des Modèles (Inst. Math., Univ. Catholique Louvain, Louvain-la-Neuve, 1975), pp. 241–308

[Che78]: Gregory Cherlin. ‘Superstable division rings’. In: *Logic Colloquium '77 (Proc. Conf., Wroclaw, 1977).* Vol. 96. Stud. Logic Foundations Math. North-Holland, Amsterdam-New York, 1978, pp. 99–111

[BCM79]: Walter Baur, Gregory Cherlin and Angus Macintyre. ‘Totally categorical groups and rings’. *J. Algebra* 57(2) (1979), pp. 407–440

[MS17]: Alexei G. Myasnikov and Mahmood Sohrabi. ‘ω-stability and Morley rank of bilinear maps, rings and nilpotent groups’. *J. Symb. Log.* 82(2) (2017), pp. 754–777

[Wag18a]: Frank Wagner. ‘Dimensional groups and fields’. Preprint. Hal 01235178. 2018

Lecture 2 – Groups of rank 1

In this lecture. We begin with basic notions: Morley degree, the descending chain condition, connectedness and connected component. Then we prove Reineke's theorem, which implies abelianity of connected groups of rank 1. Interestingly enough, involutions already play a role.

References:

- All this is extremely classical and can be found in [Borovik-Nesin, §5].
- Of course [Poizat, §1.3, §3.3] contains more.

3 Basic properties

3.1 The Degree

Lemma (and definition)**.** *If A is definable and has rank exactly n, then there is an integer d such that at most d disjoint, definable subsets $B_i \subseteq A$ have rank n.*

It is called the (Morley) degree of A.

Proof. Call *essentially insecable* a definable set X such that for any definable $Y \subseteq X$, one has $\operatorname{rk} Y < \operatorname{rk} X$ or $\operatorname{rk}(X \setminus Y) < \operatorname{rk} X$.

We prove that every definable set of rank n is a finite disjoint union of e.i. subsets of rank n. If X is a counterexample, then it is not e.i. itself: so there is $X_0 \subseteq X$ with $\operatorname{rk} X_0 = \operatorname{rk}(X \setminus X_0) = n$. Now at least one of them, and we may assume it is X_0, is not e.i.: whence $X_0 = X_{0,0} \sqcup X_{0,1}$. Iterate and construct definable $X_{0,\dots,0,1}$ all disjoint, of rank n, contained in X: against $\operatorname{rk} X = n$.

Now suppose $X = \sqcup_{i=1}^{d} X_i = \sqcup_{j=1}^{e} Y_j$ are two decompositions into e.i. subsets of rank n; we may assume $d > e$. Since $X_1 = \sqcup_{j=1}^{e}(X_1 \cap Y_j)$ is e.i., there is a unique j with $\operatorname{rk}(X_1 \cap Y_j) = n$. Iterate and find a map $\{1, \dots, d\} \to \{1, \dots, e\}$; since the Y_j are e.i., the map is injective. This is a contradiction.

So the number of e.i. components of X is well-defined: it is easily seen to be the degree. □

Remark. The e.i. components are well-defined up to the following equivalence relation on non-empty, definable sets:

$$X \sim Y \quad \text{if} \quad \operatorname{rk}(X \triangle Y) < \operatorname{rk} X,$$

which asserts that X and Y are "essentially" equal.

This would be rather anecdotical if the relation were not to play a crucial role in the elimination of groups of rank 3 during the final lecture.

Proposition. *All objects below are supposed to be definable.*

- *If A is finite then* $\deg A = |A|$.
- *If $B \subseteq A$ have the same rank, then* $\deg B \leq \deg A$.
- *If A, B are disjoint and have the same rank then* $\deg(A \cup B) = \deg A + \deg B$.
- $\deg(A \times B) = \deg A \cdot \deg B$.
- *If $f : A \to B$ and $\operatorname{rk} f(A) = \operatorname{rk} A$, then $\deg f(A) \leq \deg A$; equality holds if f is injective but the converse need not be true.*

Proof. Exercise. □

3.2 Descending chain condition

Lemma. *Let $K \leq H$ be two definable groups in a theory of finite Morley rank.*

- *If $[H : K] = \infty$, then $\operatorname{rk} H > \operatorname{rk} K$.*
- *If $[H : K] = d$, then $\deg H = d \times \deg K$.*
- *If $\operatorname{rk} H = \operatorname{rk} K$ and $\deg H = \deg K$, then $H = K$.*

Proof. Definability, the rank, the degree, are preserved by definable bijections. Among such functions are the translation maps $x \mapsto hx$ for $h \in H$. It follows that hK always has the same rank and degree as K: all claims become obvious. □

Chain conditions are common in the model-theoretic study of groups; finiteness of the Morley rank gives us the strongest possible results.

Corollary (descending chain condition)**.** *Let G be a group of finite Morley rank. Then every descending sequence of definable subgroups terminates.*

Proof. Let $(H_i)_{i\in I}$ be such a sequence. As rank and degree are ordinals, at some point the sequences $\operatorname{rk} H_i$ and $\deg H_i$ must become stationary.

So it remains to show that whenever $K \leq H$ are definable groups with same rank and degree, equality holds. This is because if H is a disjoint union of translates of K, which all have same rank and degree as K: so there is only one such, i.e. $H = K$. □

Remarks.

- As one sees, it suffices for the group to have *ordinal* Morley rank (i.e., be totally transcendental)—this is equivalent in a countable language to ω-stability.
- There is also a weaker *stable* descending chain condition: on uniformly definable subgroups. For a review of various chain conditions, see [Poizat, §1.3].

3.3 Connected component

Lemma (and definition: connected component)**.** *Let G be a group of finite Morley rank. Then there is a smallest definable subgroup of finite index. It is definably characteristic in G and called the* connected component *of G, denoted G°.*

Proof. Intersect all definable subgroups of finite index: by the descending chain condition, we get a definable subgroup, of finite index, and clearly minimal as such. It is easily seen that G° is definably characteristic in G. □

Remark. Be very careful that there is no general notion of connected components for definable sets: there is no topology here.

Definition (connected group)**.** *Let G be a group structure. Call G (model-theoretically)* connected *if it has no proper definable subgroup of finite index.*

Proposition (Cherlin)**.** *Let G be a group of finite Morley rank. Then the following are equivalent:*

- *G is connected;*
- *$G = G^\circ$;*
- *$\deg G = 1$.*

(i)⇔(ii) and (iii)⇒(i) are obvious; (i)⇒(iii) is not. Its proof (left as an exercise) requires a very useful lemma.

Lemma. *Let G be a connected group of finite Morley rank. If G acts definably on a finite set, then it fixes it pointwise.*

Proof. The stabiliser $C_G(x)$ of any point $x \in X$ is definable and has finite index: by connectedness, $C_G(x) = G$, so G fixes X pointwise. □

3.4 Minimal groups and groups of rank 1

In some sense, our exploration of the Cherlin-Zilber conjecture starts here, with the investigation of rank 1 groups. Remember the algebraic phenomenon.

Fact ([Humphreys, Theorem 20.5])**.** *Any connected, algebraic group of dimension* 1 *is one of the following:*

- $\mathbb{G}_a$*, viz. the additive group* $\mathbb{K}_+$*;*
- $\mathbb{G}_m$*, viz. the multiplicative group* $\mathbb{K}^\times$*;*
- *an elliptic curve.*

There is nothing similar for groups of finite Morley rank. In rank 1 one cannot decently expect to retrieve a field structure; as a matter of fact one cannot even expect our group to behave in the same strict way as $\mathbb{G}_a$ or $\mathbb{G}_m$ always do, for any field.

Examples. All the following have (in the pure group language) rank and degree 1:

$$\mathbb{Q}_+, \quad \mathbb{Q}_+^2, \quad \mathbb{Z}/p^\infty\mathbb{Z}, \quad (\mathbb{Z}/p^\infty\mathbb{Z})^2, \quad (\mathbb{Z}/p^\infty\mathbb{Z}) \oplus (\mathbb{Z}/q^\infty\mathbb{Z}), \quad \mathbb{Q}_+^\kappa \oplus (\mathbb{Z}/p^\infty\mathbb{Z})^a, \quad \mathbb{F}_p^\kappa, \dots$$

but *not* $\mathbb{Q}_+ \oplus \mathbb{F}_p^\kappa$.

However all examples above (including elliptic curves) share one feature: abelianity. We shall prove this; as a matter of fact the proof instantly extends to a wider class of groups, definably minimal groups of finite Morley rank.

Definition ((definably) minimal group)**.** *A group is* definably minimal *if it is infinite and has no infinite,* definable*, proper subgroup.*

Remarks.

- A definably minimal group is connected.
- (Surprisingly non-trivial) fact from geometry: the only definably minimal algebraic groups are $\mathbb{K}_+$, $\mathbb{K}^\times$, and the elliptic curves; and they happen to have dimension 1. (There is a model-theoretic proof, fairly involved as well.)
- A connected group of rank 1 is definably minimal, but the converse need not hold (there are counterexamples in model theory).

Theorem (Reineke [**Rei75**]; see the final notes)**.** *A definably minimal group of finite Morley rank is abelian.*

The proof requires a general lemma and definition of interest.

Lemma. *Let G be a connected group of finite Morley rank with a finite centre. Then $G/Z(G)$ is centreless.*

Proof. It is classical in group theory to introduce the second centre:

$$Z_2(G) = \pi^{-1}(Z(G/Z(G))) = \{g \in G : \forall h \in G\ [g,h] \in Z(G)\},$$

where $\pi : G \to G/Z(G)$ is the canonical projection and $[g,h] = g^{-1}h^{-1}gh$. We hence aim at proving that $Z_2(G) = Z(G)$. So fix $g \in Z_2(G)$ and consider the definable map:

$$\begin{array}{rccc} \mathrm{Ad}_g : & G & \to & Z(G) \\ & h & \mapsto & [g,h] \end{array}$$

In general this map is no group homomorphism; but since $g \in Z_2(G)$, it does take its values in $Z(G)$ and is multiplicative. Now $\ker \mathrm{Ad}_g$ has finite index in G, so by connectedness, $\ker \mathrm{Ad}_g = G$, which means $g \in Z(G)$, as desired. □

Definition. *A definable subset $X \subseteq G$ is* generic *if it has maximal rank.*

The notion is especially useful if G is connected (i.e. has degree 1): for then, two generic subsets must intersect.

Proof of Reineke's Theorem. Let G be a definably minimal group of finite Morley rank; we suppose that G is non-abelian and get a contradiction.

[Rei75]: Joachim Reineke. 'Minimale Gruppen'. *Z. Math. Logik Grundlagen Math.* 21(4) (1975), pp. 357–359

First, $Z(G) < G$ is definable, hence finite. By the Lemma, $G/Z(G)$ is centreless; it remains infinite, and it even remains definably minimal—but it cannot be abelian since it is centreless. So we may work towards a contradiction starting from $G/Z(G)$, which is centreless. In othe words, we may assume $Z(G) = \{1\}$.

Let $x \in G\setminus\{1\}$. Then $C_G(x) < G$, so by definable minimality it is finite. Since $x \in C_G(x)$, the order of x is finite. But also $\operatorname{rk} x^G = \operatorname{rk} G - \operatorname{rk} C_G(x) = \operatorname{rk} G$; hence x^G is generic in G. Now the same holds of any other $y \in G \setminus \{1\}$: hence any two non-trivial elements are conjugate and have finite order. So all non-trivial elements have the same *prime* order p.

Observe how x and x^{-1} are conjugate: so there is $z \in G$ with $x^{-1} = x^z$. In particular $z^2 \in C_G(x)$. If the order of z is odd, then $z \in \langle z^2 \rangle \leq C_G(x)$, so $x^{-1} = x^z = x$, and x has even order: a contradiction. So the order of z is even; being prime it is 2.

Now *all* non-trivial elements have order 2; it is well-known that G is abelian. □

Corollary. *Every infinite group of finite Morley rank contains an infinite, definable, abelian subgroup.*

Proof. Consider $H < G$ of minimal rank and degree: it is definably minimal in Reineke's sense. □

Hence a group of finite Morley rank is built from abelian blocks. Of course inside a linear algebraic group, the situation is better as the Jordan decomposition [Humphreys, §15] tells us that there are essentially two such blocks: $\mathbb{K}_+$ and $\mathbb{K}^\times$. We cannot a priori expect this to hold in our more general setting, but the situation is less hopeless than at first sight. Turning to our original question, the following is derived at once.

Corollary (cf. [Humphreys, §20.1])**.** *A connected group of rank* 1 *is abelian.*

Bear in mind that as opposed to algebraic geometry there are infinitely many isomorphism types, some not related to an algebraically closed field. So "matter" inside groups of finite Morley rank, though abelian at the atomic scale, is not always what one expects.

Final notes and exercises

Chain conditions in model theory

- The descending chain condition for groups of finite Morley rank is the strongest there is; it remains true in a group of ordinal Morley rank, i.e. totally transcendental (which in a countable language is equivalent to ω-stability).
- It is however not true in stable groups, for which one has to work with *uniformly* definable subgroups. There even is a condition which holds in the NIP context; the interested reader will see [Poizat, §1.3].
- Interestingly enough, the strong chain condition is also true in the *o*-minimal case [**Pil88**, Remark 2.13]. This is non-trivial since although *o*-minimal dimension is well-behaved, there is no analogue of the Morley degree.
- Since some form of topology plays a role in the study of *o*-minimality, one should not be surprised that our strong chain condition fails in the more general setting of finite-dimensional groups.

Connected component

- To define the connected component one needs the strongest descending chain condition; in the stable case, one merely has φ-connected components, wrt. uniformly definable families $\varphi(\cdot, \cdot)$.
  The *o*-minimal case is not stable, but there is a notion of connected component—as a matter of fact there are two, one taking the infinitesimals into account.
- Let me stress that even in finite Morley rank, things are not as trivial as they first look: let G be a group of finite Morley rank and $H_i \leq G$ be *uniformly definable* subgroups. It is open whether the H_i° are uniformy definable as well. One can show that $\deg H_i$ is bounded, a non-trivial result [**PP02**, Remark B.2.(iii)].

[Pil88]: Anand Pillay. 'On groups and fields definable in *o*-minimal structures'. *J. Pure Appl. Algebra* 53(3) (1988), pp. 239–255

[PP02]: Anand Pillay and Wai Yan Pong. 'On Lascar rank and Morley rank of definable groups in differentially closed fields'. *J. Symbolic Logic* 67(3) (2002), pp. 1189–1196

Genericity

The notion, as one expects, goes far beyond groups of finite Morley rank. For the stable case, see [Poizat, §2.1], with the following crucial fact.

Theorem (Poizat; [Poizat, Lemma 2.5])**.** *Let G be a stable group and $A \subseteq G$ a definable subset. Then A is generic in G iff finitely many translates of A cover G.*

The finite Morley rank case is an exercise below. Notice that the same phenomenon holds in *o*-minimal groups [Pil88, Lemma 2.4], where *large* means: of maximal dimension. It so appears that genericity is a group-theoretic, not a model-theoretic property. This was explored further by Poizat [**Poi14**].

On Reineke's theorem

- Reineke proved his theorem for (weakly) *minimal* groups, in the classical model-theoretic sense: every definable subset of G is finite or cofinite. This certainly implies that definable, proper subgroups are finite, but it also forces infinite conjugacy classes to intersect.
  It is unclear who first realised that definable minimality *plus* some extra model-theoretic assumption would suffice. But definable minimality of a group (a group- and model-theoretic property) is not the same as minimality of the underlying set (a purely model-theoretic property), so confusions are possible.
  - It is true that a (setwise) minimal group is abelian (this is Reineke's original version).
  - It is true that a (groupwise) minimal group of finite Morley rank is abelian (this is what we proved in class).
  - As a matter of fact, it is even true that a (groupwise) minimal, ω-stable group is abelian.
  - But it is *not* true that a (groupwise) minimal group is abelian.
    In pure group theory, there are non-abelian, infinite groups with no other proper subgroups than the cyclic p-group (Tarski monster, [Ols79]). Of course their first-order theories are disgusting.
    Shockingly enough, it is not known whether a group of finite Morley rank can contain such a monster.
- Poizat [**Poi10**] has generalised Reineke's theorem to groups of Cantor rank 1. The Cantor rank inside a given model is computed just like the Morley rank; the theory is extremely fragile as it can change when one goes to an elementary extension. So the generalisation is non-trivial.
- There are further generalisations in [**Poi09**], relaxing definable minimality taking various equivalence relations into account.
- Existence of a dimension as in [Wag18a] does not seem to be enough. However the theorem is of course true in the *o*-minimal context [Pil88, Corollary 2.15].
- Only with considerable effort could Rosengarten [Ros91] prove that a definably minimal Lie ring of finite Morley rank *of characteristic* ≥ 5 *or* 0 is abelian. The proof requires highly non-trivial material on finite Lie algebras. It is open in characteristic 3 (and rather meaningless in characteristic 2).

In all the following exercises, G is a group of finite Morley rank.

Exercise. Prove that the relation on non-empty, definable sets: $X \sim Y$ if $\text{rk}(X \triangle Y) < \text{rk}\, X$, is an equivalence relation.

Show that e.i. components are well-defined up to this relation.

Exercise. Prove the following properties left as an exercise (all objects below being definable).

- If A is finite then $\deg A = |A|$.
- If $B \subseteq A$ have the same rank, then $\deg B \leq \deg A$.
- If A, B are disjoint and have the same rank then $\deg(A \cup B) = \deg A + \deg B$.
- $\deg(A \times B) = \deg A \cdot \deg B$.
- If $f : A \to B$ and $\text{rk}\, f(A) = \text{rk}\, A$, then $\deg f(A) \leq \deg A$; equality holds if f is injective but the converse need not be true.

Exercise. If $X \subseteq G$ is any set, then $C_G(X)$ is definable.

Exercise. Let $f : G \to G$ be a definable group homomorphism. If f is injective, then it is surjective (if one does not assume that f is a homomorphism, this is a notoriously open question, presumably not simpler than the Cherlin-Zilber conjecture itself).

Exercise. Any ascending chain of definable, connected subgroups is stationary.

[**Poi14**]: Bruno Poizat. 'Supergénérix'. *J. Algebra* 404 (2014). À la mémoire d'Éric Jaligot., pp. 240–270
[**Poi10**]: Bruno Poizat. 'Groups of small Cantor rank'. *J. Symbolic Logic* 75(1) (2010), pp. 346–354
[**Poi09**]: Bruno Poizat. 'Quelques tentatives de définir une notion générale de groupes et de corps de dimension un et de déterminer leurs propriétés algébriques'. *Confluentes Math.* 1(1) (2009), pp. 111–122

Exercise. The following three are equivalent:

- G is connected;
- $G = G^\circ$;
- $\deg G = 1$.

For (i)⇒(iii), write $G = \sqcup_{i=1}^k X_i$ with $k > 1$ and $\deg X_i = 1$, and let G act on itself by left translation. Using the definability axiom, find a definable action of G on $\{1, \ldots, k\}$ and deduce that $g \cdot X_i$ intersects X_i in a full rank set.

Then consider $Y_1 = \{(x_1, x_2) \in X_1 \times X_2 : x_1 x_2 \in X_1\}$ and $Y_2 = \{(x_1, x_2) \in X_1 \times X_2 : x_1 x_2 \in X_2\}$ to contradict $\deg(X_1 \times X_2) = 1$.

Exercise. Classify connected groups of Morley rank 1 up to isomorphism.

Exercise. By the DCC, any subset $X \subseteq G$ is contained in a smallest, definable subgroup $\langle X \rangle_{\text{def}}$ called the *(definable) envelope* of X. This is particularly useful in questions such as the following.

1. A group of finite Morley rank with no involutions is uniquely 2-divisible, i.e. $\forall x\ \exists! y\ x = y^2$.
   [Hint: $\langle x \rangle_{\text{def}}$ is definable, abelian, and has no involutions.]
2. Return to the property (*) (see the exercises of Lecture 1):

   for any two involutions i, j of a *finite* group, at least one of the following holds:
   - i and j are conjugate;
   - there is an involution k commuting with both i and j.

   If G has finite Morley rank, then (*) holds.

Exercise. Prove that for definable $X \subseteq G$, one has: X is generic in G iff finitely many translates of X cover G.

Hint: this is by induction on $\text{rk}(G \setminus X)$ and $\deg(G \setminus X)$. First prove that if G is connected, $A \subseteq G$ is (definable and) generic, $B \subseteq G$ is definable, then $\{g \in G : \text{rk}(gA \cap B) = \text{rk}\, B\}$ is generic.

Lecture 3 – Groups of rank 2

In this lecture. Our non-trivial business starts here. We prove a fundamental theorem by Cherlin: every connected group of rank 2 is at most soluble. The proof of this early (1979) result is illuminating, and typical of what the domain would become: local analysis and the quest for involutions.

References:

- First appeared in [Che79, §4].
- Modern exposition in [Poizat, Theorem 3.16] or [Borovik-Nesin, Theorem 9.19].

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4 Groups of rank 2

Theorem (Cherlin). *A connected group of rank* 2 *is soluble.*

Definition (Borel subgroup). *A* Borel subgroup *is a definable, connected, soluble subgroup which is maximal as such.*

In algebraic geometry Borel subgroups play a prominent role and one understands their behaviour rather well. Inside a given connected linear algebraic group G, Borel subgroups:

- are self-normalising (i.e. $N_G(B) = B$) [Humphreys, Theorem 23.1];
- are non-nilpotent as soon as G is non-nilpotent [Humphreys, Proposition 21.4.B];
- are centreless as soon G is [Humphreys, Corollary 21.4];
- cover G [Humphreys, Theorem 22.2];
- are conjugate [Humphreys, Theorem 21.3].

In the finite Morley rank case, all one knows is that Borel subgroups are infinite almost self-normalising (i.e. $[N_G(B) : B]$ is finite). Conjugacy is out of reach in general; our study in ranks 2 and 3 will use smallness of the configurations to prove it and does not extend to arbitrary groups of finite Morley rank.

Proof. Let G be a connected group of rank 2; suppose that G is not soluble.

Step 1. *We may suppose that G is definably simple, i.e. has no non-trivial, proper, definable, normal subgroup.*

Proof. Let $1 < N \lhd G$ be one such. If N is infinite then both N° and G/N° have rank 1 so they are abelian: G is soluble, a contradiction. So N is finite; since G is connected, $N \leq Z(G)$. This also proves that $Z(G)$ is finite. Now $G/Z(G)$ is a non-soluble connected group of rank 2; as we know, it is centreless. And we just proved that it is definably simple. $\Diamond$

We now perform the so-called *local analysis* of G, i.e. the systematic study of intersections of Borel subgroups. This is motivated by the fact that nothing else is available; it tends to yield information in "small" configurations; the most ambitious notion of smallness being called $N_\circ^\circ$, and quite tougher [DJ16].

Step 2. *Borel subgroups of G are abelian, disjoint, conjugate, and have Morley rank* 1.

(Bear in mind two subgroups $H_1, H_2 \leq G$ are *disjoint* if $H_1 \cap H_2 = \{1\}$, which cannot be avoided.)

Proof. By Reineke's theorem, Borel subgroups are non-trivial; since G is non-soluble, they are proper. So they have rank 1 and by Reineke's theorem again, Borel subgroups are abelian. If $B_1 \neq B_2$ are distinct Borel subgroups meeting in $x \in B_1 \cap B_2$, then $C_G^\circ(x) \geq \langle B_1, B_2 \rangle > B_1$; hence $C_G^\circ(x) = G$ and $x \in Z(G) = 1$. This proves disjunction but conjugacy remains.

Conjugacy results in group theory usually come from counting arguments (eg. Sylow p-subgroups of a finite group) or topological methods (eg. Borel subgroups of an algebraic group); we have neither and genericity arguments will be used instead. The idea is quite simple: for any Borel subgroup B, the family of conjugates has rank 1, and each term brings a rank 1 contribution, with no overlap.

For a neat formalisation, consider the definable map:

$$f: \begin{array}{ccc} B \times G & \to & G \\ (b, g) & \mapsto & b^g \end{array}$$

The fibre above $b_0^{g_0} \neq 1$ is $\{(b, g) \in B \times G : b^g = b_0^{g_0}\}$. Such a pair must satisfy:

$$b_0 = b^{g g_0^{-1}} \in B \cap B^{g g_0^{-1}} \setminus \{1\}$$

Since Borel subgroups are disjoint, one has $B^{g g_0^{-1}} = B$, implying $g \in N_G(B) g_0$: so g remains in a rank 1 set. And once g is given, there is a unique possibility for b. So the fibre above $b_0^{g_0}$ has rank at most 1. By additivity, one finds:

$$\operatorname{rk}\left(\bigcup_g B^g\right) = \operatorname{rk} f(B \times G) \geq \operatorname{rk}(B \times G) - 1 \geq \operatorname{rk} G$$

Hence the set $B^G = \bigcup_g B^g$ is generic in G. A more conceptual approach is as follows. The definably set:

$$\mathcal{F} = \{B^g : g \in G/N_G(B)\}$$

has rank $\operatorname{rk} G - \operatorname{rk} N_G(B) = \operatorname{rk} G - \operatorname{rk} B$, with disjoint members, so:

$$\operatorname{rk}\left(B^G\right) = \operatorname{rk}\left(\bigcup \mathcal{F}\right) = \sum_{G/N_G(B)} \operatorname{rk} B^g = \operatorname{rk} G - \operatorname{rk} B + \operatorname{rk} B = \operatorname{rk} G,$$

which works only since conjugates of B are disjoint (or at least, do not intersect in the generic element of B).

In any case, B^G is generic in G. Now take another Borel subgroup $C < G$; since G is connected, it has degree 1, so the sets B^G and C^G must intersect. There are therefore $(b, c, g_1, g_2) \in B \times C \times G^2$ with $b^{g_1} = c^{g_2}$. Up to conjugating $B \cap C \neq 1$, so by disjunction, $B = C$, as desired. $\lozenge$

So far we did not only prove *generic covering* by Borel subgroups, we also have *generic partitioning* since Borel subgroups are disjoint. This is pathological from the algebraic point of view: Borel subgroups of an algebraic group (in addition to being non-nilpotent) intersect over algebraic tori.

Step 3. *For any $a \in G \setminus \{1\}$, $C_G^\circ(a)$ is a Borel subgroup containing a.*

Proof. We first prove that $C_G^\circ(a)$ is a Borel subgroup; we prove it has rank 1. Since $Z(G) = 1$, the rank is not 2; suppose for an instant that $\operatorname{rk} C_G^\circ(a) = 0$. Then the conjugacy class a^G is generic in G. But there exists one Borel subgroup B, and we proved that B^G is generic as well. So up to conjugacy $a \in B$, and by abelianity we find $B \leq C_G^\circ(a)$: a contradiction.

It remains to prove that $a \in C_G^\circ(a)$; up to conjugacy say $B = C_G^\circ(a)$. Notice that if $1 \neq x \in aB \subseteq C_G(B)$, then equality holds in $C_G^\circ(x) \geq B$. Consider the definable set:

$$X = \bigcup_{g \in G} (aB)^g;$$

we contend that X is generic. First, if $x \in (aB)^g \cap (aB)^h$ then $C_G^\circ(x) = B^g = B^h$, implying:

$$(aB)^g = a^g B^g = x \cdot B^g = x \cdot B^h = a^h B^h = (aB)^h$$

Hence distinct conjugates of aB are disjoint; on the other hand $N_G(aB) \leq N_G((aB)^{-1} \cdot (aB)) = N_G(B)$, so by additivity:

$$\operatorname{rk} X = \operatorname{rk}(G/N_G(aB)) + \operatorname{rk}(aB) = \operatorname{rk} G - \operatorname{rk} N_G(aB) + \operatorname{rk}(aB) \geq \operatorname{rk} G - \operatorname{rk} B + \operatorname{rk} B = \operatorname{rk} G,$$

and X is generic in G.

Finally there is $x \in X \cap \bigcup_{g \in G} B^g$, so up to conjugacy $x \in aB \cap B^g$. Returning to $C_G^\circ(x) = B = B^g$ we find $aB = B$ and $a \in B$. $\lozenge$

In particular we have more than generic partitioning: Borel subgroups genuinely partition G. Leaving disjunction aside, genuine covering is pathological from the finite point of view (no finite group has a covering subgroup)—though not from the algebraic point of view (as Borel subgroups always cover a connected algebraic group). Next, a real-life property.

Step 4. *Borel subgroups are self-normalising.*

Proof. Suppose $N_G(B) > B$ and let $w \in N_G(B) \setminus B$ witness this. Since $C_G^\circ(w)$ has rank 1, the conjugacy class w^G has rank and degree 1. Let $X = w^G \cap N_G(B)$ and $Y = w^G w \setminus N_G(B)$; since both are definable and partition w^G, exactly one of them is cofinite and the other is finite (possibly empty).

- If $Y = \emptyset$ then $w^G \subseteq N_G(B)$, so $\langle w^G \rangle_{\text{def}} \leq N_G(B)$. As the former is a non-trivial, proper, definable, normal subgroup, we find a contradiction. (In a simple group, a conjugacy class cannot be confined in a definable proper subgroup.)
- If Y is finite but non-empty then the connected group B which certainly normalises Y, must centralise it. So for $y \in Y$ one has $C_G^\circ(y) = B$, which is normalised by y, against the definition of Y.
- Hence Y is infinite and X is finite; since $w \in X$, it is non-empty.

Here again the connected group B must centralise the finite set X; in particular B

centralises w. So here again $B = C_G^\circ(w)$; as we know, $w \in B$, a contradiction. ◊

Step 5. *Involutions, strongly real elements, and contradiction.*

Proof. We shall create an involution using a 2-transitive action; the method is classical and worth remembering.

Let B be a Borel subgroup; the set of conjugates $X = \{B^g : g \in G/N_G(B)\}$ has Morley rank and Morley degree equal to 1. By conjugacy, G acts definably and transitively on X; in this action one has $\mathrm{Stab}_G(B) = N_G(B) = B$. Now B acts on $X \setminus \{B\}$, the set of proper conjugates. Since $\mathrm{Stab}_B(B^g) = B \cap B^g = 1$ whenever $g \notin B$, the orbit $B * [B^g] \subseteq X$ has Morley rank 1: it is generic in X. This holds of any orbit $\neq \{B\}$: so any B-orbit in $X \setminus \{B\}$ is generic in X. Since $\deg X = 1$, we deduce that $B = N_G(B) = \mathrm{Stab}_G(B)$ is transitive on $X \setminus \{B\}$. This means that G is 2-transitive on X.

Now fix another Borel subgroup $B^g \neq B$. By 2-transitivity there is $x \in G$ with $B^x = B^g$ and $B^{gx} = B$, so that $x^2 \in N_G(B) \cap N_G(B^g) = B \cap B^g = 1$; of course $x \neq 1$, so x is an involution.

With involutions one invariably studies strongly real elements. Let $i \neq j$ be distinct involutions (otherwise, $Z(G) \neq 1$) and $y = ij$ be their product. As always $y^i = y^{-1}$, so i normalises $B_y = C_G^\circ(y)$. As we know, $i \in B_y$ and $j \in B_y$ as well; as the latter is abelian, i and j commute. Hence $\left\langle i^G \right\rangle_{\mathrm{def}} \leq C_G(i)$, and G has an infinite, abelian, normal subgroup: a contradiction. ◊

This completes the proof. □

Remarks.

- If G is not nilpotent, then one can show that there is an algebraically closed field $\mathbb{K}$ with $G/Z(G) \simeq \mathrm{GA}_1(\mathbb{K}) = \mathbb{K}^\times \ltimes \mathbb{K}_+$. This uses linearisation results and is therefore not in the scope of the present course.
- One *cannot* classify nilpotent groups of rank 2. Baudisch [Bau96] has constructed $2^{\aleph_0}$, pairwise non-isomorphic, connected nilpotent groups of rank 2. None of them belongs to algebraic geometry (none defines an infinite field; however every non-abelian algebraic group does).
- Remember that not all groups of finite Morley rank are $\aleph_1$-categorical (Zilber proved it for *simple* groups; but this is not true of say $\mathrm{GL}_2(\mathbb{C})$).

  However Tanaka [**Tan88**] has proved that any non-abelian connected group of rank 2 is $\aleph_1$-categorical (not true in the abelian case).
- Rosengarten [Ros91] could prove with less difficulty that a connected Lie ring of rank 2 (and characterstic $\neq 2, 3$) is soluble.

Final notes and exercises

- Here again Poizat [Poi10] has generalised the result to groups of *Cantor* rank 2.
- A typical example of a conjugacy theorem relying on genericity argument is the conjugacy of so-called "maximal decent tori" obtained by Cherlin [**Che05**]. This completely changed our understanding of matter inside groups of finite Morley rank.

  Bear in mind that Borel subgroups are not in general conjugate. Proving it would dramatically simplify a number of cases in the Cherlin-Zilber conjecture, but one simply does not know how to attack the problem.

Exercise. Reprove Step 5 of Cherlin's rank 2 theorem by considering the left-translation action of G on G/B instead.

Exercise. For people familiar with Macintyre's theorem on abelian groups [Borovik-Nesin, Theorem 6.7]: classify connected, abelian groups of rank 2 up to isomorphism.

[Tan88]: Katsumi Tanaka. 'Nonabelian groups of Morley rank 2'. *Math. Japon.* 33(4) (1988), pp. 627–635
[Che05]: Gregory Cherlin. 'Good tori in groups of finite Morley rank'. *J. Group Theory* 8(5) (2005), pp. 613–621

Exercise. Let G be a connected, nilpotent, non-abelian group of rank 2.

1. Prove that $G' \leq Z(G)$ (admit the following consequence of the Chevalley-Zilber generation lemma: for any integer k, the subgroup G^k defined by nilpotent induction $G^{k+1} = [G^k, G]$ is definable and connected).
2. Show that there are a prime number p and $g \in G \setminus Z(G)$ such that $g^p \in Z(G)$ (hint: for $g \in G \setminus Z(G)$, there is an integer n with $g^n \in Z(G)$).
3. For $x \in G$, consider $\mathrm{Ad}_x : G \to G'$ doing $\mathrm{Ad}_x(y) = [x, y]$. Show that if $x \notin Z(G)$, then $G/C_G(x) \simeq G'$.
4. Prove that G' has exonent p (hint: $G' = \mathrm{im}\,\mathrm{Ad}_g$; now for $x \in G$, one has $[g, x]^p = [g^p, x]$, where g is as above).
5. Conclude that G has exponent at most p^2 (hint: $G/C_G(g) \simeq (G/G')/(C_G(g)/G')$).
6. Deduce Tanaka's result that any non-abelian, nilpotent group of rank 2 is $\aleph_1$-categorical.
   Hint: show that G has no Vaught pairs.

Exercise. For people familiar with Zilber's field theorem: classify connected, non-nilpotent groups of rank 2 up to isomorphism (answer: $Z(G)$ is finite and $G/Z(G) \simeq \mathrm{GA}_1(\mathbb{K}) = \mathbb{K}_+ \rtimes \mathbb{K}^\times$).

Exercise. Let $\mathfrak{g}$ be a connected Lie ring of rank 2 and characteristic $\neq 2, 3$. Prove that $\mathfrak{g}$ is soluble.
Hint: using Rosengarten's version of Reineke's theorem (which one should admit), fix a subring $\mathfrak{h} \subset \mathfrak{g}$ of rank 1. Then for $x \notin \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$, prove that $\mathfrak{g} = [x, \mathfrak{h}] \oplus \mathfrak{h}$. Deduce that $[\mathfrak{g}, \mathfrak{g}] \leq [x, \mathfrak{h}]$.

Lecture 4 – Groups of rank 3: no involutions

In this lecture. The last two lectures are devoted to proving the Cherlin-Zilber conjecture in rank 3. We shall only admit some permutation group theory.
It is a long contradiction proof whose first major step is the elimination of involutions, after Nesin. This relies on Hilbert's coordinatisation theorem in projective geometry.

References:

- First published in [**Nes89**].
- Generalisation in [**Cor89**].
- Some reference material in [Borovik-Nesin, §13] and [Poizat, §3.8], but our argument is slightly different and more general as discussed at the end of the lecture.

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5 Groups of rank 3

Theorem (Cherlin-Nesin-Frécon)**.** *A simple group of Morley rank* 3 *is isomorphic to* $\mathrm{PGL}_2(\mathbb{K})$.

Bear in mind that in the case of an algebraically closed field (and all our fields are algebraically closed), $\mathrm{PGL}_2(\mathbb{K}) \simeq \mathrm{PSL}_2(\mathbb{K})$. This is not true over over fields, for instance not true over $\mathbb{R}$. The confusion is systematic but if one thinks algebraically, illegitimate.

From now on, and for two lectures, G stands for a simple group of Morley rank 3 not isomorphic to $\mathrm{PGL}_2(\mathbb{K})$.

Proposition (Cherlin [Che79])**.** *Borel subgroups of G have rank* 1*; they are abelian, conjugate, disjoint, and they partition G.*

We must *admit* this proposition, which was later generalised by Hrushovski into a wonderful result in permutation group theory.

Fact (Corollary to [**Hru89**])**.** *Let G be a simple group of finite Morley rank with a definable subgroup of corank* 1*. Then $G \simeq \mathrm{PGL}_2(\mathbb{K})$.*

Remark. The configuration is already inconsistent, morally speaking, for several reasons.

- A connected, algebraic group with nilpotent Borel subgroups is nilpotent [Humphreys, Proposition 21.4.B].
  (Of course we are still trying to prove algebraicity.)

[Nes89]: Ali Nesin. 'Nonsolvable groups of Morley rank 3'. *J. Algebra* 124(1) (1989), pp. 199–218
[Cor89]: Luis Jaime Corredor. 'Bad groups of finite Morley rank'. *J. Symbolic Logic* 54(3) (1989), pp. 768–773
[Hru89]: Ehud Hrushovski. 'Almost orthogonal regular types'. *Ann. Pure Appl. Logic* 45(2) (1989). Stability in model theory, II (Trento, 1987), pp. 139–155

- More geometrically, any simple algebraic group is covered by conjugates of a proper subgroup (a Borel subgroup), but it cannot be *partitioned*.

  This impossibility for algebraic G is non-trivial; for finite G, it is; bear in mind that there are infinite (non-algebraic) groups G which admit such a partitioning.
- $\mathrm{PGL}_2(\mathbb{K})$ still enjoys *generic* partitioning: fix a torus T (say the diagonal matrices); then distinct conjugates T^g are disjoint, and $\coprod_G T^g$ is generic in G. *However, unipotent elements are not included.*
- One real-life group meets the above: $\mathrm{SO}_3(\mathbb{R})$. It however does not have finite Morley rank as it interprets $\mathbb{R}$ (a by-product of the proof below, but there is a two-line argument).

Proposition (Nesin)**.** *G has no involutions.*

The key idea is that if G has an involution, then it defines an incidence geometry, namely a projective 3-space [**Har66**, §2] (the definition is given in due time). Now this is coordinatisable by a celebrated theorem of Hilbert [**Hil99**] (also see [**Art57**]). Then algebraic geometry quickly finishes the problem; we do *not* invoke Bachmann's Theorem—see the final notes.

Proof. From the previous proposition we recall that Borel subgroups are conjugate, and that every $g \neq 1$ belongs to a unique Borel subgroup, $B_g = C_G^\circ(g)$.

Suppose that there is an involution.

Step 1.

(i) All involutions are conjugate.

(ii) Every Borel subgroup B contains a unique involution i,

(iii) and $C_G(i) = N_G(B) = B \rtimes \langle k \rangle$ (inversion action), where the translate kB consists of involutions.

(iv) For any two distinct involutions, there is a unique involution distinct from and commuting with both.

Remarks.

- All four hold in $\mathrm{SO}_3(\mathbb{R})$.
- In our notation, both sets $k^G \cap N_G(B)$ and $k^G \setminus B$ are infinite; the former equals kB and has rank 1; the latter has rank 2. This should be compared with the proof of self-normalisation in rank 2.

Proof. Let i be an involution. Since $i \in B_i$ and Borel subgroups are conjugate, every Borel subgroup has an involutions; for (ii) we need uniqueness.

(i) Let i and j be *non-commuting* involutions, so that the strongly real element $x = ij \neq 1$ is not an involution. Remember that i inverts x, since:

$$x^i = ixi = i \cdot ij \cdot i = ji = j^{-1}i^{-1} = (ij)^{-1} = x^{-1}.$$

In particular i normalises $C_G^\circ(x) = B_x$; we see two cases. If $B_x = B_i$, then i centralises B_x, so it centralises $x \in B_x$; on the other hand i inverts x, so $x^{-1} = x$ and $x^2 = 1$, a contradiction. Therefore $B_x \neq B_i$, so $B_x \cap B_i = \{1\}$. It is not hard to see that i inverts B_x.

Notice that B_x does not have exponent 2 (otherwise G does, and is then abelian); so B_x is 2-divisible, and there is $y \in B_x$ with $y^2 = x$. Now i also inverts $y \in B_x$, so:

$$i^y = y^{-1}iy = y^i iy = iyiiy = iy^2 = ix = iij = j,$$

so i and j are conjugate.

We finally cover the case where i and j commute. There must be an involution k commuting to neither (otherwise, $\left\langle i^G \right\rangle \leq C_G(i,j)$, a contradiction), and now i, k, j are conjugate.

[**Har66**]: Robin Hartshorne. *Foundations of projective geometry.* Vol. 1966/67. Lecture Notes, Harvard University. W. A. Benjamin, Inc., New York, 1967. vii+167 pages
[**Hil99**]: David Hilbert. *Grundlagen der Geometrie.* Leipzig: Teubner, 1899. 92 pages
[**Art57**]: Emil Artin. *Geometric algebra.* Interscience Publishers, Inc., New York-London, 1957. x+214 pages

(ii) We actually proved that every Borel subgroup B is inverted by some involution $k \in N_G(B) \setminus B$.

Observe that *any* involution $k \in N_G(B) \setminus B$ inverts B. Otherwise $C_B(k)$ is infinite, so $B \cap B_k \neq 1$; by disjunction $k \in B_k = B$, a contradiction.

In particular, if $\ell \in N_G(B) \setminus B$ is another involution, then $k\ell$ centralises B, implying $B_{k\ell} = B$; hence $k\ell \in B$ and $\ell \in kB$. So all involutions in $N_G(B) \setminus B$ are equal modulo B.

We can now prove that there is a unique involution in B. Bear in mind that it does not have exponent 2, so it has only finitely many involutions; let $V < B$ be the finite group they generate. There exists at least one k inverting B, so k centralises V; conversely V normalises B_k. Now for $i \in V \setminus \{1\}$ one has $i \notin B_k$ since otherwise $k \in B_k = B_i$; this implies that i inverts B_k.

So all involutions in V invert B_k: there can be only one since if $i \neq j$ are in $V \setminus \{1\}$, then $1 \neq ij$ is yet another involution in V, which must both invert and centralise B: a contradiction.

(iii) We return to the normaliser $N_G(B)$. Of course $C_G(i) \leq N_G(C_G^\circ(i)) = N_G(B)$; now if $\sigma \in N_G(B)$ then σ must centralise the unique involution $i \in B$, so one has $N_G(B) = C_G(i)$.

It remains to show that $N_G(B) \setminus B$ consists of involutions; we know that they are all equal modulo B. Let $\sigma \in N_G(B) \setminus B = C_G(i) \setminus B$; the unique involution $i \in B$ centralises σ, so it normalises B_σ. If i centralises B_σ then $\sigma \in B_\sigma = B_i = B$, a contradiction. So i inverts B_σ and also $\sigma \in B_\sigma$: so σ is an involution, and we are done.

(iv) Let $i \neq j$ be involutions. If i and j commute, then $k = ij$ is an involution distinct from both which commutes with them. By the structure of the Sylow 2-subgroup, it is the only such.

From now on we suppose that i and j do not commute. Then $x = ij$ is *not* an involution. Since i and j invert x, they normalise $C_G^\circ(x)$, so they centralise the unique involution there. If conversely ℓ is an involution commuting with both i and j, then ℓ centralises x. In particular ℓ normalises $C_G^\circ(x)$, but it cannot invert it since otherwise $x = x^\ell = x^{-1}$. Hence $\ell \in C_G^\circ(x)$ is the only involution there. ◊

We introduce a geometry. Call *points* the elements of G. Call *lines* the elements of $\Lambda = \{gBh : (g,h) \in G^2\} = \{gB^h : (g,h) \in G^2\}$. Call *planes* the elements of $\Pi = \{gI : g \in I\}$. Our incidence relation is set-theoretic; in particular it is both left- and right-covariant.

Step 2. (G, Λ, Π) *is a projective* 3*-space.*

Proof. Following [Har66, §2] there are six axioms.

1. First axiom: any two distinct points lie on a unique line.

   Group-theoretic translation: for $x \neq y \in G$, there is a unique translate of a Borel subgroup gC containing both x and y. Alternative translation: every non-trivial element lies in a unique Borel subgroup. This is just partitioning G by its Borel subgroups.

   Clearly $xB_{x^{-1}y}$ is a line through x and y; it is the only such since $gC = xC = yC$ implies $x^{-1}y \in C$.

2. Bonus axiom one and a half: if a plane contains two distinct points, then it contains the line through them.

   Let $x \neq y$ be the two points lying on P; we may assume $y = 1$; then $P = jI$ for some involution j. Since $x \in jI$, the involution j inverts x (and $j \neq x$). The line through 1 and x is the Borel subgroup B_x, which is normalised by j. But $j \notin B_x$: if x is an involution, then it is the only such in B_x while $j \neq x$; if x is not an involution, then $j \notin C_G(x)$ so $j \notin B_x$. In any case j inverts B_x, meaning $B_x \subseteq jI$.

3. Bonus axiom almost two: three non-collinear points define a unique plane.

   Up to translating we may suppose the points are $1, x, y$; non-collinearity means that

x and y do not lie in the same Borel subgroup. Let i_x be the unique involution in B_x, i_y likewise; then $i_x \neq i_y$ so there is a unique involution k distinct from both and lying in both. Now k normalises both B_x and B_y but is in none, so it inverts them; in particular it inverts x and y. Hence $1, x, y \in kI$. Clearly any plane containing 1 must be of the form ℓI, and clearly only k meets the requirements.

4. Second axiom: any line and non-incident point lie on a unique plane.

   Take two distinct points on the line; this gives a non-collinear triple (x, y, z). There is a unique plane containing it, which also contains the line.

5. Third axiom: a line and a plane are always incident.

   There are good reasons not to prove this one (see the exercises).

6. Fourth axiom: any two planes share at least a line.

   There are good reasons not to prove this one (see the exercises).

7. Fifth axiom: there are four points no three of which are collinear.

   Obvious.

8. Sixth axiom: every line has at least three points.

   Obvious. ♢

Every projective 3-space is arguesian (namely, every projective plane it contains satisfies Desargues's property: this is the classical proof of Desargues' theorem), and as such, coordinatisable by [Hil99, §§24-26]. The more elegant and conceptual proof of [Art57, §§2.3–2.4, 2.6] is reproduced in [Har66, §7].

So there is a skew-field $\mathbb{K}$ with $G \simeq \mathbb{P}^3(\mathbb{K})$. Importantly, $\mathbb{K}$ is definable (Hilbert's construction is explicit; this is harder to see with Artin's proof), so it actually is an infinite skew-field of finite Morley rank. By early work of Cherlin-Shelah-Macintyre [CS80], [Mac71], $\mathbb{K}$ is an algebraically closed (commutative) field.

Step 3. *Contradiction.*

Proof. G acts definably on itself by left translations; as we know the action preserves incidence, so we get a definable homomoprhism $G \to \mathrm{Aut}(\mathbb{P}^3(\mathbb{K}))$. Since fields of finite Morley rank have no definable groups of automorphisms, we actually have a homomorphism $G \to \mathrm{PGL}_4(\mathbb{K})$. Since G is simple, this is a definable group embedding. We shall conclude with a little algebraic geometry.

The Zariski closure $\beta = \overline{B} \leq \mathrm{PGL}_4(\mathbb{K})$ is easily seen to remain abelian and connected (now in the geometric sense); by Borel's fixed point theorem [Humphreys, Theorem 21.2], β must have a fixed point when acting on the complete variety $\mathbb{P}^3(\mathbb{K})$; hence B has a fixed point.

But the left action of G on itself is the regular representation: no element is fixed by a non-trivial element. ♢

This removes involutions from the picture. □

The residual configuration is thus highly pathological from the perspective of finite group theory, which is the realm of the Feit-Thompson theorem.

Final notes and exercises

Historical notes

In his paper (§5.2, Theorem 1.(4)) Cherlin already claimed that there are no involutions. The proof is not correct, as observed by Nesin in his PhD; however he gives credit to Cherlin in his own work.

The lack of involutions is essentially equivalent to self-normalisation of the Borel subgroups, something a priori hard to obtain; Cherlin seeks the latter. The flaw in his argument is in the next-to-last sentence of the next-to-last paragraph of his proof (on p. 25 of his article): "Thus $A^\circ \cap C(x) \neq 1$, hence $A^\circ = C(x)^\circ$". In our notation, x is an involution inverting the Borel subgroup A°; there would be an involution $i \in A^\circ$ which is inverted, hence centralised, by x. This entails $i \in A^\circ \cap C(x)$, but it is *not* enough to jump to $A^\circ \cap C(x)^\circ \neq 1$. It is true that Borel subgroups are disjoint, and therefore so are *connected components* of centralisers, but centralisers themselves could a priori fail to be disjoint.

Borovik had reached a similar conclusion; it was published as a communication to the Soviet Academy of Sciences in 1984, and is now hard to locate.

On Bachmann's theorem

Both [Borovik-Nesin] and [Poizat] rely on a final contradiction different from ours.

Theorem (Bachmann-Nolte [**Nol79**]-Schröder [**Sch82**]; see [Borovik-Nesin, Fact 8.15])**.** *Let G be a group. Call three involutions $i, j, k \in G$* collinear *if ijk is an involution. Suppose that with respect to this relation, the set of involutions I is a projective plane.*

Then the group generated by I is isomoprhic to $\mathrm{SO}_3(\mathbb{K}, f)$, for $\mathbb{K}$ a definable field of characteristic $\neq 2$ and f a non-isotropic quadratic form on $\mathbb{K}^3$.

Returning to the rank 3 configuration, one studies the set of involutions and sees the assumptions are satisfied; hence Bachmann's theorem applies. Since a definable field must be algebraically closed by Macintyre's theorem, every quadratic form on $\mathbb{K}^n$ is isotropic: this is how [Borovik-Nesin] and [Poizat] derive their contradiction (Nesin's original strategy is slightly different).

We cannot agree methodologically. Bachmann and his school were after an algebraic axiomatisation of so-called "metric geometry", i.e. geometry where one has both an incidence *and* an orthogonality relations. This is a magnificent completion of the Hilbert axiomatisation; the keystone of the millennial cathedral of geometry, the *chorus mysticus* concluding the Euclidean poem. Everything in Bachmann's work in conducted in terms of abstract groups generated by involutions: the primary concept is that of a reflection (involution) and orthogonality is a property of the product of two involutions. The title of Bachmann's book is *Aufbau der Geometrie aus dem Spiegelungsbegriff*. One cannot be surprised, in presence of an orthogonality relation, to have a result producing both a field *and* a quadratic form.

But metric-like structures, even in the meek geometric setting of quadratic forms, cannot appear in finite Morley rank nor even in the stable case. Which suggests that there ought to be a simpler proof—if not conceptually, then at least culturally.

Finally observe that the contradiction usually given relies on the geometry of I, whereas we studied the global geometry of G. (Interestingly, Nesin's original argument [Nes89] did introduce the 3-dimensional geometry). Of course G acts on I by conjugacy; however the action is not regular as involutions do have non-trivial centralisers. In short, focusing on the mere plane of involutions one has to drop regularity and the final argument from algebraic geometry.

Around arguesianity

Another issue with restricting the geometry to I is that although it clearly is a projective plane (at least in our notion), it could fail to be arguesian. It is well-known that every projective 3-space is arguesian; it is not true of a projective plane. As a matter of fact Baldwin [**Bal94**] could construct non-arguesian projective planes of finite Morley rank. So focusing on I actually gives one extra work.

Bad groups

The absence of involutions generalises to a whole class of (hypothetical) pathological configurations, the ill-named *bad groups*—since Cherlin could not expect the impact of his 1979 article he certainly cannot be blamed for introducing the terminology.

One should bear in mind two things on bad groups: first, the definition has varied over the years; second, so did the name. It is not quite unreasonable to suggest the following terminology:

- a *paradoxical Frobenius group* is a group G with a proper subgroup H whose conjugates partition G;
- an *asomic group* is a group G which cannot define an infinite field.

We know from Zilber's work that in an asomic group of finite Morley rank, Borel subgroups must be nilpotent (this is sometimes called the "Field Theorem", but actually a corollary also involving Chevalley-Zilber generation by indecomposable sets; see [Borovik-Nesin, Corollary 9.10]). By the Borovik-Cherlin-Corredor-Poizat study [Borovik-Nesin, Theorem 13.3], a simple, asomic group of finite Morley rank must be a paradoxical Frobenius group, where H is a Borel subgroup. Nesin in rank 3, then Corredor in any rank, proved that such a group has no involutions; it quickly follows that Borel subgroups are self-normalising.

Baudisch [Bau96] constructed non-abelian asomic groups, but they are nilpotent. Whether a simple asomic group exists is completely open, but would directly refute the Cherlin-Zilber conjecture. By the way, the free group is being proved asomic these years (so far only Cartesian definable field, i.e. fields definable as subsets of G^n, could be ruled out [**BS15**]).

[Nol79]: Wolfgang Nolte. 'Gruppen mit Involutionen, welche Quadriken bestimmen'. *Arch. Math. (Basel)* 33 (1979), pp. 177–182

[Sch82]: Eberhard Schröder. 'Eine gruppentheoretisch-geometrische Kennzeichnung der projektiv-metrischen Geometrien'. *J. Geom.* 18(1) (1982), pp. 57–69

[Bal94]: John Baldwin. 'An almost strongly minimal non-Desarguesian projective plane'. *Trans. Amer. Math. Soc.* 342(2) (1994), pp. 695–711

[BS15]: Ayala Byron and Rizos Sklinos. 'Fields definable in the free group'. Preprint. arXiv:1512.07922. 2015

Further generalisations

Theorem (D.-Wiscons [**DW18a**, Corollary A]). *Let G be a connected group of finite Morley rank and $C < G$ be a definable, connected subgroup such that $G = \bigsqcup_{g \in G/N_G(C)} C^g$. Then G has no involutions.*

(Notice that one does *not* suppose C to be self-normalising; as a matter of fact one cannot prove it a posteriori, only in special cases.)

The proof begins with the thorough analysis of the Sylow 2-subgroup; this is fairly classical but not in the scope of the present course. Then one uses exactly the argument we gave.

The above generalisation may look fairly optimal; and yet Wiscons and myself tend to interpret it as the emerging tip of something much more general, an identification result of $\mathrm{PGL}_2(\mathbb{K})$ among simple groups of finite Morley rank with involutions, under *generic* assumptions. This would also explain why $\mathrm{SO}_3(\mathbb{R})$ is everywhere in this lecture: it is just another so-called *real form* of $\mathrm{PGL}_2(\mathbb{K})$, meaning it has the same root system, because it has the same involutive configuration. This is research at work, joint with Wiscons.

Exercise. Prove, in but a few lines, that $\mathrm{SO}_3(\mathbb{R})$ defines the field $\mathbb{R}$.

Hint. On the set of involutions (which are the half-turns of the spaces, i.e. $\mathbb{P}^2(\mathbb{R})$), consider the relation: $i \varepsilon j$ if ($[i,j] = 1$ and $i \neq j$). Letting L_i be the axis of i, show that $i \varepsilon j$ iff $L_i \perp L_j$. Deduce that $(I, \varepsilon) \simeq \mathbb{P}^2(\mathbb{R})$.

Call *Cherlin-Nesin configuration* (with or without involutions) a non-algebraic, simple group of rank 3. Nesin has proved that such a configuration has no involutions and Frécon, that it is outright inconsistent. But we use neither in the following exercises. What we *do* use it that Borel subgroups have rank 1 (Cherlin-Hrushovski).

Exercise. Prove that in the Cherlin-Nesin configuration, Borel subgroups are the $C_G^\circ(a)$ for $a \in G \setminus \{1\}$, and that they partition G. (Hint: just follow the rank 2 study.)

Exercise. In the Cherlin-Nesin configuration *with* involutions, prove that $G = I \cdot B$.

Exercise. In the Cherlin-Nesin configuration *with* involutions, prove the missing axioms directly. Then solve the following.

Exercise. In the Cherlin-Nesin configuration *with* involutions, let $g^\perp = gI$. Prove that this extends to a well-defined polarity of the incidence structure (one must in particular give the definition of the dual of a plane, and of a line). Check that incidence, and left- and right-covariance are preserved. Conclude that it is enough to prove four out of the six axioms to obtain the projective 3-space structure.

Exercise. In the Cherlin-Nesin configuration *with* involutions, prove that three involutions i, j, k are collinear iff ijk is an involution.

Exercise. In the Cherlin-Nesin configuration *without* involutions, prove that Borel subgroups are self-normalising.

Lecture 5 – Groups of rank 3: the end

In this lecture. Proof of the Cherlin-Zilber conjecture in rank 3.

References:

- Initially proved in [**Fré18**].
- First rewriting (still in geometric terms) in [**PW16**].
- The notion of symmetry is borrowed from [**Poi18**].
- The main line follows [**Wag18b**].

Yet the approach given here (joint with Corredor) manages to entirely remove any form of reference to geometry, which we deem artificial.

[**DW18a**]: Adrien Deloro and Joshua Wiscons. 'The Geometric Theorem (Paris Album No.1)'. In preparation. 2018

[**Fré18**]: Olivier Frécon. 'Simple groups of Morley rank 3 are algebraic'. *J. Amer. Math. Soc.* 31(3) (2018), pp. 643–659

[**PW16**]: Bruno Poizat and Frank Wagner. 'Comments on a Theorem by Olivier Frécon'. Preprint. arXiv 1609.06229 (Modnet 1095). 2016

[**Poi18**]: Bruno Poizat. 'Milieu et symétrie, une étude de la convexité dans les groupes sans involutions'. *J. Algebra* 497 (2018), pp. 143–163

[**Wag18b**]: Frank Wagner. 'Bad groups'. In: *Mathematical Logic and its Applications*. Ed. by Makoto Kikuchi. Vol. 2050. RIMS Kôkyûroku. Kyoto: Kyoto University, 2017, pp. 57–66

Let G be a simple group of rank 3 *not* isomorphic to $\mathrm{PGL}_2(\mathbb{K})$; we know that it has no involutions. It is easy and classical (yet necessary for the final contradiction) to also remove definable involutive *outer automorphisms.*

Proposition (Delahan-Nesin, [Borovik-Nesin, Proposition 13.4])**.** *G has no involutive definable automorphism.*

All this is morally unbearable: of which we must convince the group as well.

Proposition (Frécon)**.** *Contradiction.*

Proof. What follows is the shortening obtained with Corredor of the Wagner rewriting of the proof by Poizat-Wagner of Frécon's Theorem on the Nesin configuration.

For the final contradiction we shall obtain an involutive automorphism by constructing its graph; actually a "large" fragment of the graph suffices. Here the notion of size is purely group-theoretic.

Definition (confined set)**.** *Call a definable set* confined *if it is included in a finite union of translates of definable, proper subgroups.*

Since we know that there are no definable subgroups of rank 2, every confined set has rank at most 1.

In order to construct the graph of an involutive automorphism, one needs to argue in $G \times G$ (where graphs live naturally) and to add as an extra symmetry the possibility to swap coordinates (swapping certainly is an involution). This should motivate the following.

Definition ($*$-bi-G-set)**.** *A* $*$-bi-G-set *is a $G \times G$-set Ω equiped with an involutive bijection $* : \Omega \to \Omega$ subject to:*

$$\forall (a, b, \omega) \in G \times G \times \Omega \quad ((a, b) \cdot \omega)^* = (b, a) \cdot \omega^*$$

Here are a few examples:

- G, for the left- and right-action and inversion as $*$, i.e. $(a, b) \cdot g = agb^{-1}$ and $g^* = g^{-1}$;
- $\mathcal{P}(G)$, for $(a, b) \cdot X = aXb^{-1}$ and $X^* = X^{-1} = \{x^{-1} : x \in X\}$;
- $\mathcal{P}_{def}(G)$, same structure.

  Now both the action and the star are compatible with the equivalence relation on non-empty sets: $X \sim Y$ if $\mathrm{rk}(X \triangle Y) < \mathrm{rk}\, X$. (We set $\mathrm{rk}\, \emptyset = -\infty$ and add $[\emptyset]_\sim = \{\emptyset\}$ to prevent zerological questions.)
- $\Omega = \mathcal{P}_{def}(G)/\sim$ is therefore another $*$-bi-G-set. One may check, if necessary and in obvious notation:

$$((a, b) \cdot \omega)^* = (a[X]b^{-1})^* = [(aXb^{-1})^{-1}] = [bX^{-1}a^{-1}] = b[X]^*a^{-1} = (b, a) \cdot \omega^*$$

These examples are *not* definable. But the stabiliser inside $G \times G$ of any element of $\mathcal{P}_{def}(G)/\sim$ is, which will suffice. Also notice that the only $G \times G$-fixed points in $\mathcal{P}_{def}(G)/\sim$ are $[\emptyset]$ and $[G]$.

The following definition is after Poizat.

Definition (symmetry)**.** *For $g \in G$ consider the definable map:*

$$\sigma_g(\omega) = (g, g^{-1}) \cdot \omega^*.$$

Clearly σ_g is an involutive bijection of Ω since:

$$\sigma_g^2(\omega) = (g, g^{-1}) \cdot ((g, g^{-1}) \cdot \omega^*)^* = (g, g^{-1}) \cdot (g^{-1}, g) \cdot \omega^{**} = \omega$$

The proof will exploit these symmetries. The key lemma has no model-theoretic assumptions.

Lemma. *Let G be a simple, 2-divisible group with no involutions, and Ω be $*$-bi-G-set with definable stabilisers. Suppose that there is $\omega \in \Omega \setminus \mathrm{Fix}(G \times G)$ such that $\Sigma(\omega) = \{g \in G : \sigma_g(\omega) = \omega\}$ is not confined. Then G has an involutive, definable automorphism.*

Proof. We shall construct the graph of the desired automorphism as a subgroup of $\mathbb{G} = G \times G$. It will be useful to let $G_1 = G \times \{1\}$ and $G_2 = \{1\} \times G$, while $\pi_i : \mathbb{G} \to G$ will denote the standard projections.

We first reduce to the case where $1 \in \Sigma(\omega)$. Fix any $g_0 \in \Sigma(\omega)$ and (by 2-divisibility) some $h_0 \in G$ with $h_0^2 = g_0$; let $\omega' = (h_0^{-1}, h_0) \cdot \omega$. We claim that $\Sigma(\omega') = h_0^{-1}\Sigma(\omega)h_0^{-1}$. Indeed one has:

$$\begin{aligned} g \in \Sigma(\omega') &\text{ iff } \sigma_g(\omega') = \omega' \\ &\text{ iff } (g, g^{-1}) \cdot (\omega')^* = \omega' \\ &\text{ iff } (g, g^{-1}) \cdot ((h_0^{-1}, h_0) \cdot \omega)^* = (h_0^{-1}, h_0) \cdot \omega \\ &\text{ iff } (h_0, h_0^{-1})(g, g^{-1})(h_0, h_0^{-1}) \cdot \omega^* = \omega \\ &\text{ iff } h_0 g h_0 \in \Sigma(\omega) \end{aligned}$$

In particular $\Sigma(\omega')$ remains unconfined, but now it contains $h_0^{-1} g_0 h_0^{-1} = 1$. Of course ω' is not fixed by $\mathbb{G}$. So we may assume that $1 \in \Sigma(\omega)$, and for simplicity we shall just write Σ.

Let $\mathbb{H} = \mathrm{Stab}_{\mathbb{G}}(\omega) \leq \mathbb{G}$, a definable subgroup. Since $1 \in \Sigma$ one has $(1,1) \cdot \omega^* = \omega$, i.e. $\omega^* = \omega$; in particular $g \in \Sigma$ iff $(g, g^{-1}) \in \mathbb{H}$, so $\Sigma \subseteq \pi_i(\mathbb{H})$. Moreover for $(a, b) \in \mathbb{G}$ one has:

$$\begin{aligned} (a, b) \in \mathbb{H} &\text{ iff } (a, b) \cdot \omega = \omega \\ &\text{ iff } (a, b) \cdot \omega^* = \omega^* \\ &\text{ iff } ((b, a) \cdot \omega)^* = \omega^* \\ &\text{ iff } (b, a) \cdot \omega = \omega \\ &\text{ iff } (b, a) \in \mathbb{H} \end{aligned}$$

We say that $\mathbb{H}$ is *swap-invariant*. Now let us prove that it is the graph of an involutive automorphism.

Since $\Sigma \subseteq \pi_i(\mathbb{H})$ is not confined, $\pi_i(\mathbb{H}) = G$. Moreover $G_i \lhd \mathbb{G}$, so $\mathbb{H} \cap G_i \trianglelefteq \mathbb{H}$ and $\pi_i(\mathbb{H} \cap G_i) \trianglelefteq \pi_i(G_i) = G$; therefore either $\pi_i(\mathbb{H} \cap G_i) = \{1\}$ or $\pi_i(\mathbb{H} \cap G_i) = G$. In the latter case, since $\mathbb{H}$ is swap-invariant one finds $\mathbb{H} = \mathbb{G}$: then ω is $\mathbb{G}$-fixed, a contradiction.

As a conclusion $\pi_i(\mathbb{H} \cap G_i) = \{1\}$, i.e. $\mathbb{H}$ is the graph of an injective functional relation. Since $\pi_i(\mathbb{H}) = G$, the relation is everywhere defined and surjective; since $\mathbb{H} \leq \mathbb{G}$ is a definable subgroup, the relation is actually a definable automorphism α of G. Finally since $\mathbb{H}$ is swap-invariant, α has order at most 2; now observe that α inverts $\Sigma = \{g \in G : (g, g^{-1}) \in \mathbb{H}\}$. Since G has no involutions, α has order exactly 2. □

We return to the proof. We shall find a non-trivial $\omega \in \Omega = \mathcal{P}_{def}(G)/\sim$ with unconfinedly many symmetries, which will contradict the Delahan-Nesin Proposition.

Let $\beta : \mathbb{G} \to G$ be the commutator map, $1 \neq c_0 \in \beta(\mathbb{G})$ any fixed commutator, and:

$$X = \pi_1(\beta^{-1}(\{c_0\})) = \{a \in G : \exists b \in G : [a, b] = c_0\}.$$

This is our candidate. It is a clever observation (which Poizat made in his kitchen) that $[X]_\sim \neq [G]_\sim$; this is done immediately hereafter. The problem will then be to determine the set of symmetries of X (or rather, $[X]_\sim$); there will be an unexpected paragraph as $\deg X$ remains a mystery.

Step 1. *X is not generic in G.*

Proof. A priori, for one given $a_0 \in X$ the set $\{b \in G : [a_0, b] = c_0\}$ is parametrised by a translate of $C_G(a_0)$, so it has rank 1; it follows that $\beta^{-1}(\{c_0\}) = \pi_1^{-1}(X) \subseteq \mathbb{G}$ has rank $1 + \mathrm{rk}\, X$. This is a conjugacy-invariant property, i.e. for any $g \in G$ one has $\mathrm{rk}\, \beta^{-1}(\{c_0^g\}) = 1 + \mathrm{rk}\, X$. Since $\mathrm{rk}(c_0^G) = \mathrm{rk}\, G - 1$ we find:

$$\mathrm{rk}\, \beta^{-1}(c_0^G) = \mathrm{rk}\, G - 1 + 1 + \mathrm{rk}\, X = \mathrm{rk}\, G + \mathrm{rk}\, X$$

So if X is generic in G then $U = \beta^{-1}(c_0^G)$ is generic in $\mathbb{G}$.

Now $U^{\leftrightarrow}$ obtained by swapping coordinates is generic again; since $\deg \mathbb{G} = 1^2 = 1$, so is $U \cap U^{\leftrightarrow}$. Then for (a, b) in a generic subset of $\mathbb{G}$, both $g = [a, b]$ and $[b, a] = [a, b]^{-1}$ are conjugate to c_0, hence between themselves. So there is x with $g^x = g^{-1}$ and x^2 centralises g. Since $x \in \langle x^2 \rangle_{\text{def}} \leq C_G(g)$, we find $g^2 = 1$, a contradiction. ◊

Step 2. *A point with unconfinedly many symmetries—contradiction.*

Proof. The idea is that X is almost its set of symmetries (making $\Sigma([X]_\sim)$ a fortiori unconfined). Recall that for $g \in G$, $B_g = C_G^\circ(g)$ is the only Borel subgroup of G containing g.

For $a_0 \in X$ let:

$$Y_{a_0} = \bigcup_{b \in B_{a_0} b_0} B_b a_0.$$

Observe that if $B_b a_0 \cap B_{b'} a_0 \neq \emptyset$, then $B_b = B_{b'}$ and $B_b a_0 = B_{b'} a_0$: the union is disjoint. It is indexed by a rank 1 family, so $\operatorname{rk} Y_{a_0} = 2 = \operatorname{rk} X$. Finally if $a \in Y_{a_0}$, say $a \in B_b a_0$ with $b \in B_{a_0} b_0$, one has:

$$[a, b] = [a_0, b] = [a_0, b_0] = c_0,$$

so that $Y_{a_0} \subseteq X$. Both have rank 2 but one cannot jump to $Y_{a_0} \sim X$ since the latter has unknown degree. On the other hand it is easy to see that $\deg Y_{a_0} = 1$. Write $X = X_1 \sqcup \cdots \sqcup X_d$ in obvious notation; then up to permuting the X_i, for infinitely many a_0 one has $Y_{a_0} \sim Y_{a_0} \cap X_1 \sim X_1$.

Also observe:

$$\sigma_{a_0}(Y_{a_0}) = a_0 Y_{a_0}^{-1} a_0 = Y_{a_0}.$$

Hence for $a \in X_1 \subseteq X$,

$$\sigma_{a_0}(X_1) \sim \sigma_{a_0}(Y_{a_0}) = Y_{a_0} \sim X_1.$$

Let $\omega = [X_1]_\sim$; by the above, $\Sigma(\omega)$ contains X_1, so it has rank ≥ 2; $\Sigma(\omega)$ is therefore unconfined. Now $\omega \neq [\emptyset]_\sim, [G]_\sim$, so one has $\omega \notin \operatorname{Fix}(\mathbb{G})$: the Lemma applies and G has an involutive definable automorphism, a final contradiction. ◊

Here has the proof an end. □

Final notes and exercises

- Poizat and Wagner actually proved the following.

  Theorem ([PW16]). *Let G be a simple, asomic group of Morley rank $2n+1$, with abelian Borel subgroups. Then these have rank $< n$.*

- In *Cantor* rank 3 this is not written yet; it is unclear whether the field $\mathbb{K}$ in $\mathrm{PGL}_2(\mathbb{K})$ should be algebraically closed. The question is a bit artificial anyway.
- Rosengarten [Ros91] had easily obtained that a connected *Lie ring* of rank 3 is of the form $\mathfrak{sl}_2(\mathbb{K})$. It would be interesting to read the argument again in light of the one for groups. And needless to say, it would be interesting to push further.
- Wiscons [**Wis16**] proved that a simple group of rank 4 would have Borel subgroups of rank 1; then Wiscons and myself [**DW18b**], proved that a simple group of rank 5 would have Borel subgroups of rank 1 or 2, but in the latter case they would be nilpotent and non-abelian.

  These were obtained before Frécon's theorem, and of course people want to remove these configurations (bear in mind there are no simple algebraic groups of Zariski-dimension 4 or 5).
- In rank 4 Wiscons' methods are rather elementary. But for rank 5 one had to call elaborate results [DJ16] belonging to the core of the Borovik programme in so-called odd type—where model theory is decidely looking towards finite group theory.

All exercises use the following fact: if a definable quotient H/K has 2-torsion, then so does H.

Exercise. Prove that the Cherlin-Nesin configuration *without* involutions has no involutive definable automorphism.

Hint: let a α be one such. Find an element centralised by α and conjugated to an element inverted by α; then get an involution.

[Wis16]: Joshua Wiscons. 'Groups of Morley rank 4'. *J. Symb. Log.* 81(1) (2016), pp. 65–79
[DW18b]: Adrien Deloro and Joshua Wiscons. 'Simple groups of Morley rank 5 are bad'. *J. Symb. Log.* 83(3) (2018), pp. 1217–1228

Exercise. Let G be a connected group of rank $2n$ with no involutions. Suppose that all proper subgroups are abelian. Prove that a proper subgroup has rank $\leq n$.

Hint: remember how an involution was produced in rank 2.

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