Logics preserving degrees of truth and the hierarchies of abstract algebraic logic

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Logics preserving degrees of truth

1. In general
2. In ordered sets

Abstract algebraic logic and its two hierarchies of logics

Logics preserving degrees of truth from residuated lattices
(substructural logics and mathematical fuzzy logic)

1. Presentation and basic facts
2. Classification
3. The Deduction Theorem
4. Completeness
Preservation of truth in (sentential) many-valued logic

\( \varphi_1, \ldots, \varphi_n \vdash \psi \iff \text{whenever all } \varphi_i \text{ are true, } \psi \text{ is true.} \)

The traditional many-valued setting:

- A set \( \mathcal{V} \) of truth values \(|\mathcal{V}| > 2\) (Łukasiewicz: \( \mathcal{V} \subseteq [0,1] \subseteq \mathbb{R} \))
- A subset \( D \subseteq \mathcal{V} \) of designated truth values (often \( D = \{1\} \))
- A set \( \text{Val} \) of evaluations \( v: \text{Language} \rightarrow \mathcal{V} \)

\( \varphi_1, \ldots, \varphi_n \vdash \psi \iff \text{for all } v \in \text{Val}, \)

\[ \text{if } v(\varphi_i) \in D \text{ for all } i = 1, \ldots, n, \text{ then } v(\psi) \in D \]

«truth comes in degrees» «truth of a fuzzy proposition is a matter of degree»
Preservation of truth when truth comes in degrees

- A **degree of truth** can be any $D \subseteq \mathcal{V}$

- A **structure of degrees of truth** on a set of truth values $\mathcal{V}$ is any family $\{D_j : j \in J\}$ with $D_j \subseteq \mathcal{V}$

$\varphi_1, \ldots, \varphi_n \models \psi \iff$ for all $v \in Val$, for all $j \in J$,

if $v(\varphi_i) \in D_j$ for all $i = 1, \ldots, n$, then $v(\psi) \in D_j$

- No individual $D_j$ has a meaning concerning truth. Only the global structure has. No need to designate one among them.

- We call this the **consequence preserving degrees of truth**
  (with respect to this structure)
Preservation of truth in ordered algebras

- $\mathcal{V}$ is an algebra $A = \langle A, \ldots \rangle$ of the type of $Fm$ (the algebra of formulas)
  - There is an order $\leq$ on $A$
  - Each truth value $a \in A$ determines a degree of truth

$$\uparrow a := \{ b \in A : a \leq b \} \subseteq A$$

- The logic that preserves this structure of degrees of truth is

$$\varphi_1, \ldots, \varphi_n \vdash_A^\leq \psi \iff \text{for all } v \in Val, \text{ for all } a \in A, \text{ if } a \leq v(\varphi_i) \text{ for all } i = 1, \ldots, n, \text{ then } a \leq v(\psi)$$

- When $A$ has a lattice reduct determining $\leq$, then

$$\varphi_1, \ldots, \varphi_n \vdash_A^\leq \psi \iff \text{for all } v \in Val, v(\varphi_1) \land \cdots \land v(\varphi_n) \leq v(\psi)$$

$$\emptyset \vdash_A^\leq \psi \iff \text{for all } v \in Val, v(\psi) = \max A$$

- Scott, Cleave, Pavelka, Nowak, Wójcicki, Gil, Hájek, Baaz
- Dunn, Belnap, Fitting, Avron, Shramko, Wansing
Abstract algebraic logic

- Henkin-Monk-Tarski (1985)
  *Cylindric algebras, Part II* (Studies in Logic ..., vol. 115)
  Section 5.6: *Abstract algebraic logic and more algebraic logics*
  «algebraic version of abstract model theory»

- General, abstract study of algebraic semantics of sentential logics

  Blok, Pigozzi; Czelakowski; Herrmann; Font, Jansana; Raftery; Jónsson; Galatos, Tsinakis; Gil-Férez; Cintula, Noguera; ...

- Workshop on abstract algebraic logic (Barcelona, 1997)


- Mathematics Subject Classification (2010 revision) code 03G27

- Two hierarchies of logics: The Leibniz hierarchy and the Frege hierarchy
The Leibniz hierarchy (main fragment)

implicative
↓
finitely regularly algebraizable

finitely algebraizable
→
regularly algebraizable

finitely equivalential

algebraizable

equivalential

weakly algebraizable

protoalgebraic

regularly weakly algebraizable
The Leibniz hierarchy: some bridge theorems

- Beth’s definability ⇐⇒ Epimorphisms are surjective: equivalental logics
  \[ \text{[Blok, Hoogland (2006)]} \]

- Deduction Theorem ⇐⇒ EDPRC: finitary, finitely algebraizable logics

- Deduction Theorem ⇐⇒ Join-semilattice of compact theories is dually residuated: finitary, protoalgebraic logics
  \[ \text{[Czelakowski (1985)]} \]

- Interpolation ⇐⇒ Amalgamation: equivalental logics

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Czelakowski, J. and Pigozzi, D.

Amalgamation and interpolation in abstract algebraic logic.

The Frege hierarchy

- Protoalgebraic + Fregean + has theorems $\implies$ regularly algebraizable
- For finitary, weakly algebraizable logics, the Frege hierarchy reduces to selfextensional and Fregean.
- Hierarchies are \textbf{orthogonal} to one another: A logic can be in the highest level of one hierarchy and totally outside the other.
(Commutative and Integral) Residuated Lattices

Definition

A **residuated lattice** is an algebra \( A = \langle A, \wedge, \vee, \star, \rightarrow, 1, 0 \rangle \) such that

1. \( \langle A, \wedge, \vee \rangle \) is a lattice with maximum 1 (with order \( \leq \)),
2. \( \langle A, \star, 1 \rangle \) is a **commutative** monoid with unit 1,
3. \( \langle A, \star, \rightarrow \rangle \) is a residuated pair with respect to the order \( \leq \), i.e.,

\[
c \star a \leq b \iff c \leq a \rightarrow b \quad \forall a, b, c \in A.
\]

The class \( RL \) of these algebras is a **variety**.

- Also denoted in the literature by \( FL_{ei} \). [Full LAMBEK calculus …]
- Some well-known **subvarieties**: \( FL_{ew}, BL, MTL, MV, \Pi, G, HA, BA, \ldots \)

**NOTATION:** equations \( \varphi \approx \psi \); order relations \( \varphi \preceq \psi := \varphi \wedge \psi \approx \varphi \)
Logics preserving truth

For $K \subseteq RL$:

- We turn truth-functional: $Val = \text{Hom}(Fm, A)$
- The designated truth set in each $A \in K$ is $\{1\}$

**Definition**

$L_K$ is the finitary logic determined by:

$$
\varphi_1, \ldots, \varphi_n \vdash_K \psi \iff K \models \varphi_1 \approx 1 \& \ldots \& \varphi_n \approx 1 \rightarrow \psi \approx 1
$$

$$
\emptyset \vdash_K \psi \iff K \models \psi \approx 1
$$

Informally, we call $L_K$ the logic of 1 (associated with $K$)

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**Galatos, N.; Jipsen, P.; Kowalski, T.; Ono, H.**

*Residuated lattices: an algebraic glimpse at substructural logics*

Definition

For $K \subseteq RL$, $\vdash^K$ is the finitary logic determined by:

$$\varphi_1, \ldots, \varphi_n \vdash^K \psi \iff K \models \varphi_1 \land \cdots \land \varphi_n \preceq \psi$$

$$\emptyset \vdash^K \psi \iff K \models \psi \approx 1$$

Informally, we call $\vdash^K$ the logic of order (associated with $K$)

- The logic $\vdash^K$ depends only on the equations satisfied by $K$.
- $\varphi \vdash^K \psi \iff K \models \varphi \approx \psi$

For an arbitrary class $K \subseteq RL$, $\vdash^K = \vdash^{V(K)}$

So $K$ will be a subvariety of $RL$ from now on.

Bou, Esteva, Font, Gil, Godo, Torrens, Verdú
Logics preserving degrees of truth from varieties of residuated lattices

Font Taking degrees of truth seriously


Font, Gil, Torrens, Verdú
On the infinite-valued Łukasiewicz logic that preserves degrees of truth

Font An abstract algebraic logic view of some multiple-valued logics
The logics $\vdash_K$ and $\vdash_{\leqslant}^K$ have the same theorems or tautologies.

$\varphi_1, \ldots, \varphi_n \vdash_K \leqslant \psi \iff \emptyset \vdash_K (\varphi_1 \land \cdots \land \varphi_n) \to \psi$

$\iff \emptyset \vdash_K (\varphi_1 \land \cdots \land \varphi_n) \to \psi$

$\varphi \vdash_K \leqslant \psi \iff \emptyset \vdash_K \varphi \to \psi$ ["graded" DT]

The logic $\vdash_K$ is the inferential extension of $\vdash_{\leqslant}^K$ by any of:

The rule of Modus Ponens for $\to$: $\varphi, \varphi \to \psi \vdash \psi$

The rule of Adjunction for $\star$: $\varphi, \psi \vdash \varphi \star \psi$

The rule of $\star$-squaring: $\varphi \vdash \varphi \star \varphi$

If $A \in K$ then $\mathcal{F}_i\vdash_K (A) = \{\text{implicative filters of } A\}$.

If $A \in K$ then $\mathcal{F}_i\vdash_{\leqslant}^K (A) = \{\text{lattice filters of } A\}$.

**Notation:** $\vdash$ separates the two sides of rules or sequents (no $\vdash$ nor $\Rightarrow$)
Classification in the hierarchies

- For each $K$, the logic $\vdash_K$ is **implicative**, with

  equivalent algebraic semantics $\text{Alg}(\vdash_K) = K$,
  equivalence formula $x \leftrightarrow y := (x \to y) \star (y \to x)$, and
  defining equation $x \approx 1$,

  but need not be even selfextensional.

- For each $K$, the logic $\vdash_{\leq K}$ is **fully selfextensional**, with

  $\text{Alg}(\vdash_{\leq K}) = \mathcal{V}(\vdash_{\leq K}) = K$ (the **intrinsic variety**),

  but need not be even protoalgebraic.
Classification in the hierarchies (II)

For each $K$, the following conditions are equivalent:

1. The logic $\vdash^\leq K$ is Fregean.
2. The logic $\vdash^\leq K$ is weakly algebraizable.
3. The logic $\vdash_K$ is selfextensional.

4. The two logics coincide: $\vdash_K = \vdash^\leq K$. Equivalently, any of:
   1. $\star$ coincides with $\land$ in $K$.
   2. $K$ is a variety of (generalized) Heyting algebras.
   3. $\vdash_K$ is an axiomatic extension of JOHANSSON’s minimal logic.

When this happens, the unique logic is both implicative and fully Fregean.

- Coincidence of $\star$ and $\land$ means we have left the properly substructural landscape. So it is perhaps more interesting to see what happens when $\vdash^\leq K$ belongs to lower levels of the Leibniz hierarchy.
Classification in the hierarchies (III)

For each $K$, the following conditions are equivalent:

1. The logic $\vdash^\leq K$ is protoalgebraic.

2. The logic $\vdash^\leq K$ is equivalential.

3. The logic $\vdash^\leq K$ is finitely equivalential. Then $(x \to y)^n \star (y \to x)^n$, for some $n \in \omega$, works as equivalence formula.

4. $K \models x \land ((x \to y)^n \star (y \to x)^n) \preceq y$, for some $n \in \omega$.

3 mutually excluding locations for $\vdash^\leq K$ in the Leibniz hierarchy:

- Non-protoalgebraic
- Finitely equivalential but not algebraizable
- Implicative (and then also: fully Fregean)
Some consequences

- If $\vdash^\leq_K$ is protoalgebraic, then there is $n \in \omega$ such that $K \models x^n \approx x^{n+1}$.
  
  (all algebras in $K$ are $n$-contractive)

- For most of best-known fuzzy logics ($\mathcal{L}_\infty$, BL, MTL, FL$_{ew}$, $\Pi$, etc.):
  the logic preserving degrees of truth is not protoalgebraic,
  the logic preserving truth is not selfextensional.

- If $K$ is a variety of MTL-algebras, then
  
  $\vdash^\leq_K$ is protoalgebraic $\iff$ all chains in $K$ are ordinal sums of simple $n$-contractive MTL-chains, for some $n \in \omega$.

- If $K$ is a variety of BL-algebras, then
  
  $\vdash^\leq_K$ is protoalgebraic $\iff$ there is $n \in \omega$ such that $K \models x^n \approx x^{n+1}$.

- The logic preserving degrees of truth with respect to any finite BL-chain is protoalgebraic, hence finitely equivalental.
  
  The $\mathcal{L}_n$ are finitely equivalental but not algebraizable.
Deduction Theorems for $\vdash_K$

• All the $\vdash_K$ satisfy the **Local Deduction Theorem** for $\{x^n \to y : n \in \omega\}$:

$$\Gamma, \alpha \vdash_K \beta \iff \text{there is some } n \in \omega \text{ such that } \Gamma \vdash_K \alpha^n \to \beta.$$  

[GALATOS, ONO (2006); POGORZELSKI (1964) for $\mathcal{L}_\infty$]

• A logic $\vdash_K$ satisfies the **Deduction Theorem (DT)** for some $\delta(x, y)$:

$$\Gamma, \alpha \vdash_K \beta \iff \Gamma \vdash_K \delta(\alpha, \beta)$$

if and only if all algebras in $K$ are $n$-contractive, for some $n$.

Moreover  $\delta(x, y) = x^n \to y$.

[GALATOS (2003); POGORZELSKI (1964), WÓJCICKI (1973) for $\mathcal{L}_n$]
For each $K$, the following conditions are equivalent:

1. The logic $\vdash^\leq_K$ satisfies the DT for $\delta(x, y) = x \rightarrow y$.
2. The logic $\vdash_K$ satisfies the DT for $\delta(x, y) = x \rightarrow y$.
3. The logics $\vdash_K$ and $\vdash^\leq_K$ coincide.

If $K$ is a variety of MTL-algebras, then

$\vdash^\leq_K$ satisfies the DT for some $\delta(x, y) \iff \vdash^\leq_K$ is protoalgebraic.
Moreover $\delta(x, y) = (x \rightarrow y)^n \lor y$.

[GIL, TORRENS, VERDÚ (1993) for subvarieties of MV]

If $K$ is generated by a finite $A$, then

$\vdash^\leq_K$ satisfies the DT for some $\delta(x, y) \iff$ the following hold:

1. The logic $\vdash^\leq_K$ is protoalgebraic.
2. $A$ is distributive as a lattice.
3. $\land$ is residuated and every subalgebra of $A$ is closed under the map $\langle a, b \rangle \mapsto \max \{ c \in A : a \land c \leq b \}$.
Axiomatizations: Gentzen-style, Tarski-style

- **Sequents**: $\Gamma \vdash \varphi$ where $\Gamma$ is a finite set of formulas
- **All structural axioms and rules**, and the **logical axioms**:
  \[
  \begin{align*}
  \emptyset \vdash 1 & & \varphi, \psi \vdash \psi \\
  \emptyset \vdash \varphi \rightarrow \varphi & & \varphi \land \psi \vdash \varphi \\
  \varphi \rightarrow (\psi \rightarrow \zeta) \vdash \psi \rightarrow (\varphi \rightarrow \zeta) & & \varphi \land \psi \vdash \psi
  \end{align*}
  \]

- **Logical rules**:
  \[
  \begin{align*}
  \begin{array}{c}
  \varphi \vdash \zeta \\
  \psi \vdash \zeta
  \end{array} & \quad \begin{array}{c}
  \emptyset \vdash \varphi \rightarrow \psi
  \end{array} & \quad \begin{array}{c}
  \varphi \vdash \psi
  \end{array}
  \end{align*}
  \]

  \[
  \begin{align*}
  \begin{array}{c}
  \varphi \lor \psi \vdash \zeta
  \end{array} & \quad \begin{array}{c}
  \varphi \star \psi \vdash \zeta
  \end{array}
  \end{align*}
  \]

\[
\begin{align*}
\varphi_1, \ldots, \varphi_n \vdash_R \psi & \iff \varphi_1, \ldots, \varphi_n \vdash \psi \text{ is derivable in this calculus}
\end{align*}
\]
Axiomatizations: Hilbert-style

1. **Axioms**: all $\varphi$ such that $K \models \varphi \approx 1$ (i.e., all theorems of $\vdash_K$)

2. **Rules**: Adjunction for $\land$: $\varphi, \psi \vartriangleright \varphi \land \psi$
   
   Restricted Modus Ponens: $\varphi \vartriangleright \psi$ when $\varphi \to \psi$ is a theorem

Assume $\vdash_K$ is axiomatized by $\text{Ax}(K)$, with MP as the only rule.

Then $\vdash_{\leq}^K$ is axiomatized by 1 as the **only axiom**, and the **rules**:

- $K$-specific rule: $\alpha \vartriangleright \alpha \ast \varphi$ for every $\varphi \in \text{Ax}(K)$
- Adjunction for $\land$: $\varphi, \psi \vartriangleright \varphi \land \psi$
- Modus Ponens for $\ast$: $\alpha \ast (\varphi \ast (\varphi \to \psi)) \vartriangleright \alpha \ast \psi$
- Weakening for $\ast$: $\varphi \ast \psi \vartriangleright \varphi$
- Associativity for $\ast$: $(\varphi \ast \psi) \ast \zeta \vartriangleright \varphi \ast (\psi \ast \zeta)$
- Commutativity for $\ast$: $\varphi \ast \psi \vartriangleright \psi \ast \varphi$
Some conclusions

- There may be sound reasons to consider **logics preserving degrees of truth** in the framework of many-valued logic.

- **Logics preserving degrees of truth from varieties of residuated lattices** form an **interesting** and remarkably **diverse** family of logics, with a richer, less uniform and less predictable behaviour than logics preserving truth.

- **Abstract algebraic logic** provides a powerful and convenient framework to study both particular logics and large groups of logics.

- Locating a logic in the hierachies gives information on which algebraic properties to look for in order to prove certain metalogical theorems (and conversely).

- Having **the hierachies** is important because, along the history of mathematics, having **good classifications** has been an important driving force in the development of many areas.
Some references on abstract algebraic logic

Czelakowski, J. *Protoalgebraic logics*  
vol. 10 of *Trends in Logic - Studia Logica Library*.  

Font, J. M., and Jansana, R.  
*A general algebraic semantics for sentential logics. Second revised edition*  
Electronic version freely available through Project Euclid at  

Font, J. M., Jansana, R., and Pigozzi, D. *A survey of abstract algebraic logic*  
*Studia Logica* (Special issue on Abstract Algebraic Logic) 74 (2003), 13–97.  
With an update in vol. 91 (2009), 125–130.

Cintula, P., and Noguera, C.  
*A general framework for Mathematical Fuzzy Logic*  
In *Handbook of Mathematical Fuzzy Logic, vol. I*, P. Cintula, P. Hájek, and  