HILBERT SPACES WITH GENERIC PREDICATES

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Abstract. We prove the existence of model companions for three kinds of generic expansions of Hilbert spaces: first we add a distance function to a random substructure, then a distance to a random subset and finally a random predicate. The theory obtained with the random substructure is ω-stable, while those obtained with the distance to a random subset and the random predicate are unstable. In addition to providing these model companions, we start the model theoretic study of these generic expansions.

1. Introduction

This paper deals with Hilbert spaces expanded with random predicates in the framework of continuous logic as developed in [4, 5]. The model theory of Hilbert spaces is very well understood, see [4, Chapter 15] or [7]. In section 2 we briefly review some of its properties.

There are several papers that deal with generic expansions of Hilbert spaces. Ben-Yaacov, Usvyatsov and Zadka studied the expansion of a Hilbert space with a generic automorphism. The models of this theory are the expansions of Hilbert spaces with a unitary map whose spectrum is S^1. A model of this theory can be constructed by amalgamating together the collection of n-dimensional Hilbert spaces with a unitary map whose eigenvalues are the n-roots of unity as n varies in the positive integers. More work on generic automorphisms can be found in [6], where the first author of this paper studies Hilbert spaces expanded with a random group G of automorphisms.

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There are also several papers about expansions of Hilbert spaces with random subspaces. In [7] Berenstein and Buechler identified the saturated models of the theory of beautiful pairs of a Hilbert space. An analysis of lovely pairs (the generalization of beautiful pairs to simple theories) in the setting of compact abstract theories is carried out in [2]. In the third section of this paper we build the beautiful pairs (belles paires) of Hilbert spaces using the Fraïssé amalgamation method. We provide an axiomatization for this class and we show that the projection operator into the subspace is interdefinable with a predicate for the distance to the subspace. We also prove that the theory of beautiful pairs of Hilbert spaces is ω-stable (in [3] it is shown that such a theory is supersimple). Many of the properties of beautiful pairs of Hilbert spaces are known from the literature or folklore, so this section is mostly a compilation of results.

In the third section we add a predicate for the distance to a random subset. This construction was inspired by the idea of finding an analogue to the first order generic predicates studied in [8]. The axiomatization we found for the model companion was inspired in the ideas of [8] together with the following observation: in Hilbert spaces there is a definable function that measures the distance between a point and a model. We prove that the theory of Hilbert spaces with a generic predicate is unstable. We also study a natural notion of independence in a monster model of this theory and prove some of its properties, although the question of the simplicity of the model companion remains open.

Finally, in the fourth section we deal with expansions of Hilbert spaces with a random predicate that satisfies a 1-Lipschitz modulus of uniform continuity. We find a model companion for such a theory. Again the main tool used in the axiomatization was the existence of a definable function that measures the distance between a point and a model. We also study a natural notion of independence in a monster model of this theory and prove some of its properties. As before the theory obtained is unstable but it is unclear if the model companion is a simple theory.

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2. Model theory of Hilbert spaces

2.1. **Hilbert spaces.** We follow [4] in our study of the model theory of a Hilbert space and its expansions. We assume the reader is familiar with the basic concepts of continuous logic as presented in [4, 5]. A Hilbert space $\mathcal{H}$ can be seen as a multi-sorted structure $(B_n(H), 0, +, \langle , \rangle, \{\lambda_r : r \in \mathbb{R}\})_{0<n<\omega}$, where $B_n(H)$ is the ball of radius n, $+$ stands for addition of vectors (defined from $B_n(H) \times B_n(H)$ into $B_{2n}(H)$), $\langle , \rangle \colon B_n(H) \times B_n(H) \to [-n^2, n^2]$ is the inner product, 0 is a constant for the zero vector and $\lambda_r \colon B_n(H) \to B_{n(\llbracket |r| \rrbracket+1)}H$ is the multiplication by $r \in \mathbb{R}$.

We denote by L the language of Hilbert spaces and by T the theory of Hilbert spaces.

By a universal domain $\mathcal{H}$ of T we mean a Hilbert space $\mathcal{H}$ which is κ-saturated and κ-strongly homogeneous with respect to types in the language L, where κ is a cardinal larger than $2^{\aleph_0}$. Constructing such a structure is straightforward —just consider a Hilbert space with an orthonormal basis of cardinality at least κ.

We will assume that the reader is familiar with the notions of *definable closure* and *non-dividing.* The reader can check [4, 7] for the definitions.

2.1. **Notation.** Let dcl stand for the definable closure and acl stand for the algebraic closure in the language L.

2.2. **Fact.** *Let* $A \subset \mathcal{H}$ *be small. Then* $\operatorname{dcl}(A) = \operatorname{acl}(A) =$ *the smallest Hilbert subspace of* $\mathcal{H}$ *containing* A.

Proof. See Lemma 3 in [7, p. 80] □

2.2. **Non-dividing in T.** First let us recall a characterization of non-dividing in pure Hilbert spaces:

2.3. **Proposition.** *Let $B, C \subset H$ be small, let $(a_1, \ldots, a_n) \in H^n$ and assume that $C = \mathrm{dcl}(C)$, so C is a Hilbert subspace of H. Denote by P_C the projection on C. Then $\mathrm{tp}(a_1, \ldots, a_n/C \cup B)$ does not divide over C if and only if for all $i \leq n$ and all $b \in B$, $a_i - P_C(a_i) \perp b - P_C(b)$.*

Proof. Proved as Corollary 2 and Lemma 8 of [7, pp. 81–82]. □

For $A, B, C \subset H$ small, we say that A is independent from B over C if for all $n \geq 1$ and $\bar{a} \in A^n$, $\mathrm{tp}(\bar{a}/C \cup B)$ does not divide over C.

In particular, types over sets are stationary and non-dividing is *trivial*, that is, for all sets B, C and tuples $(a_1, \ldots, a_n)$ from H, $\mathrm{tp}(a_1, \ldots, a_n/C \cup B)$ does not divide over C if and only if $\mathrm{tp}(a_i/B \cup C)$ does not divide over C for $i \leq n$.

3. Random subspaces

First we deal the case of a Hilbert space with a projection operator to a subspace. Let $L_p = L \cup \{P\}$ where P is a new unary function and we consider structures of the form $(\mathcal{H}, P)$, where $P\colon H \to H$ is a projection into a subspace. Note that $P\colon B_n(H) \to B_n(H)$ and that P is determined by its action on $B_1(H)$. Recall that projections are bounded linear operators, characterized by two properties:

(1) $P^2 = P$
(2) $P^* = P$

The second condition means that for any $u, v \in H$, $\langle P(u), v\rangle = \langle u, P(v)\rangle$. A projection also satisfies, for any $u, v \in H$, $\|P(u) - P(v)\| \leq \|u - v\|$. In particular, it is a uniformly continuous map and its modulus of uniform continuity is $\Delta_P(x, y) = \|x - y\|$.

We start by showing that the class of Hilbert spaces with projections has the free amalgamation property:

3.1. **Lemma.** *Let* $(\mathcal{H}_0, P_0) \subset (\mathcal{H}_i, P_i)$ *where* $i = 1, 2$ *and* $H_1 \downarrow_{H_0} H_2$ *be (possibly finite dimensional) Hilbert spaces with projections. Then* $H_3 = \mathrm{span}\{H_1, H_2\}$ *is a Hilbert space and* $P_3(v_3) = P_1(v_1) + P_2(v_2)$ *is a well defined projection, where* $v_3 = v_1 + v_2$ *and* $v_1 \in H_1$*,* $v_2 \in H_2$*.*

Proof. It is clear that $H_3 = \mathrm{span}\{H_1, H_2\}$ is a Hilbert space containing H_1 and H_2. It remains to prove that P_3 is a projection map and that it is well defined. We denote by P_{H_1}, P_{H_2}, P_{H_3} the projections into the spaces H_1, H_2 and H_3 respectively.

Let $v_3 \in H_3$. Let $u_0 = P_{H_0}(v_3)$, $u_1 = P_{H_1}(v_3) - u_0$, $u_2 = P_{H_2}(v_3) - u_0$. Then $v_3 = u_0 + u_1 + u_2$. If $v_3 = v_1 + v_2$, with arbitrary $v_1 \in H_1$ and $v_2 \in H_2$.

Claim: There are $w_0, t_0 \in H_0$ such that $v_1 = u_1 + w_0$, $v_2 = u_2 + t_0$ and $w_0 + t_0 = u_0$.

We prove that $w_0 \in H_0$. As $v_3 - v_2 = v_1 = u_1 + w_0$, taking P_{H_1} on both sides and using that $H_2 \downarrow_{H_0} H_1$, we get $P_{H_1}(v_3) - P_{H_0}(v_2) = u_1 + w_0$, so $u_1 + u_0 - P_{H_0}(v_2) = u_1 + w_0$. Simplifying, we get $w_0 = u_0 - P_{H_0}(v_2)$, which of course means $w_0 \in H_0$. We get $t_0 \in H_0$ in a similar way.

Then $P_3(v_3) = P_1(v_1) + P_2(v_2) = P_1(u_1 + w_0) + P_2(u_2 + t_0) = P_1(u_1) + P_1(w_0) + P_2(u_2) + P_2(t_0) = P_1(u_1) + P_0(w_0) + P_0(t_0) + P_2(u_2) = P_1(u_1) + P_0(u_0) + P_2(u_2)$.

Since the expression on the right only depends on v_3 and not on the decomposition of v_3 in terms of v_1 and v_2, we get that P_3 is a well defined linear map on H_3.

Since for any $v_3 \in H_3$, $P_3(v_3) = P_1(v_1) + P_2(v_2) = P_1^*(v_1) + P_2^*(v_2) = P_3^*(v_3)$, we get that $P_3^* = P_3$.

Finally, for any $v_3 \in H_3$, $P_3^2(v_3) = P_3(P_1(v_1) + P_2(v_2)) = P_1(P_1(v_1)) + P_2(P_2(v_2)) = P_1^2(v_1) + P_2^2(v_2) = P_1(v_1) + P_2(v_2) = P_3(v_3)$, so $P_3^2 = P_3$.

□

Given an n-dimensional Hilbert space $\mathcal{H}_n$, there are only $n+1$ many pairs $(\mathcal{H}_n, P)$ modulo isomorphism. They are classified by the dimension of $P(H)$, which ranges from 0 to n.

We can consider the age $\mathcal{K}_P$ formed by the pairs $(\mathcal{H}_n, P)$, where $\mathcal{H}_n$ is a Hilbert space of dimension n; following the usual Fraïssé construction, we get a Hilbert space $\mathcal{H}_\omega$ of dimension $\aleph_0$, equipped with a projection P_ω, in which we can elementarily embed any finite-dimensional pair $(\mathcal{H}, P)$. Clearly, $P_\omega(\mathcal{H}_\omega)$ and its orthogonal complement $(P_\omega(\mathcal{H}_\omega))^\perp$ both have dimension $\aleph_0$. In particular, $P_\omega(H_\omega)$ is a model of the theory of infinite dimensional Hilbert spaces and $\mathcal{H}_\omega$ is ω-saturated over $P_\omega(H)$. This pair obtained is the unique separable model of the theory of beautiful pairs associated to a Hilbert space. The theory T^P_ω extending T and stating that P is a projection and that there are infinitely many orthonormal vectors v satisfying $P_\omega(v) = v$ and infinitely many orthonormal vectors u satisfying $P_\omega(u) = 0$ gives an axiomatization for $Th(\mathcal{H}_\omega, P_\omega)$, the theory of beautiful pairs of Hilbert spaces.

Let $(\mathcal{H}, P) \models T^P_\omega$ and for any $v \in H$ let $d_P(v) = \|v - P(v)\|$. Then $d_P(v)$ measures the distance between v and the subspace $P(H)$. The pair $(\mathcal{H}, d_P)$ is definable from $(\mathcal{H}, P)$. We will now prove the converse.

3.2. **Lemma.** *Let $(\mathcal{H}, P) \models T^P_\omega$. For any $v \in H_\omega$ let $d_P(v) = \|v - P(v)\|$. Then $P(v) \in dcl(v)$ in the structure $(\mathcal{H}, d_P)$.*

Proof. Note that $P(v)$ is the unique element x in $P(H)$ satisfying $\|v - x\| = d_P(v)$. Thus $P(v)$ is the unique realization of the statement $\varphi(x) = \max\{d_P(x), |\|v - x\| - d_P(v)|\} = 0$. □

3.3. **Proposition.** *Let $(\mathcal{H}, P) \models T^P_\omega$. For any $v \in H_\omega$ let $d_P(v) = \|v - P(v)\|$. Then the function $P(x)$ is definable in the structure $(\mathcal{H}, d_P)$*

Proof. Let $(\mathcal{H}, P) \models T^P_\omega$ be κ-saturated for $\kappa > \aleph_0$ and let $d_P(v) = \|v - P(v)\|$. Since d_P is definable in the structure $(\mathcal{H}, P)$, the new structure

$(\mathcal{H}, d_P)$ is still κ-saturated. Let $\mathcal{G}_P$ be the graph of the function P. Then by the previous lemma $\mathcal{G}_P$ is type-definable in $(\mathcal{H}, d_P)$ and thus by [4, Proposition 9.22] P is definable in the structure $(\mathcal{H}, d_P)$. $\square$

3.4. **Notation.** We write tp for L-types, tp_P for L_P-types and qftp_P for quantifier free L_P-types. We write acl_P for the algebraic closure in the language L_P respectively. We follow a similar convention for dcl_P.

3.5. **Lemma.** *T_ω^P has quantifier elimination.*

Proof. It suffices to show that quantifier free L_P-types determine the L_P-types. Let $(\mathcal{H}, P) \models T_\omega^P$ and let $\bar{a} = (a_1, \dots, a_n), \bar{b} = (b_1, \dots, b_n) \in H^n$. Assume that $\mathrm{qftp}_P(\bar{a}) = \mathrm{qftp}_P(\bar{b})$. Then

$$\mathrm{tp}(P(a_1), \dots, P(a_n)) = \mathrm{tp}(P(b_1), \dots, P(b_n))$$

and

$$\mathrm{tp}(a_1 - P(a_1), \dots, a_n - P(a_n)) = \mathrm{tp}(b_1 - P(b_1), \dots, b_n - P(b_n)).$$

Let $H_0 = P(H)$ and let $H_1 = H_0^\perp$, both are then infinite dimensional Hilbert spaces and $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Let $f_0 \in Aut(\mathcal{H}_0)$ satisfy $f_0(P(a_1), \dots, P(a_n)) = (P(b_1), \dots, P(b_n))$ and let $f_1 \in Aut(\mathcal{H}_1)$ be such that $f_1(a_1 - P(a_1), \dots, a_n - P(a_n)) = (b_1 - P(b_1), \dots, b_n - P(b_n))$. Let f be the automorphism of $\mathcal{H}$ induced by by f_0 and f_1, that is, $f = f_0 \oplus f_1$. Then $f \in Aut(\mathcal{H}, P)$ and $f(a_1, \dots, a_n) = (b_1, \dots, b_n)$, so $\mathrm{tp}_P(\bar{a}) = \mathrm{tp}_P(\bar{b})$. $\square$

Coordinatization: By the previous lemma, the L_P-type of an n-tuple $\bar{a} = (a_1, \dots, a_n)$ inside a structure $(\mathcal{H}, P) \models T_\omega^P$ is determined by the type of its projections $\mathrm{tp}(P(a_1), \dots, P(a_n), a_1 - P(a_1), \dots, a_n - P(a_n))$. In particular, we may regard $(\mathcal{H}, P)$ as the direct sum of the two independent pure Hilbert spaces $(P(H), +, 0, \langle, \rangle)$ and $(P(H)^\perp, +, 0, \langle, \rangle)$. Thus there is a strong kind of coordinatization for our structures.

We may therefore characterize definable and algebraic closure, as follows.

3.6. **Proposition.** *Let* $(\mathcal{H}, P) \models T_\omega^P$ *and let* $A \subset H$. *Then* $\mathrm{dcl}_P(A) = \mathrm{acl}_P(A) = \mathrm{dcl}(A \cup P(A))$.

We leave the proof to the reader. Another consequence of the coordinatization is:

3.7. **Proposition.** *The theory* T_ω^P *is* ω*-stable.*

Proof. Let $(\mathcal{H}, P) \models T_\omega^P$ be separable and let $A \subset H$ be countable. Replacing $(\mathcal{H}, P)$ for $(\mathcal{H}, P) \oplus (\mathcal{H}, P)$ if necessary, we may assume that

$P(H) \cap dcl_P(A)^\perp$ is infinite dimensional and that $P(H)^\perp \cap dcl_P(A)^\perp$ is infinite dimensional. Thus every L_p-type over A is realized in the structure $(\mathcal{H}, P)$ and $(S_1(A), d)$ is separable. □

3.8. **Proposition.** *Let* $(\mathcal{H}, P) \models T_\omega^P$ *be a* κ*-saturated domain and let* $A, B, C \subset H$ *be small. Then* $\mathrm{tp}_P(A/B \cup C)$ *does not fork over* C *if and only if* $\mathrm{tp}(A \cup P(A)/B \cup P(B) \cup C \cup P(C))$ *does not fork over* $C \cup P(C)$.

Again the proof is straightforward.

4. Random subsets

We now study the expansion of a Hilbert space with a distance function to a subset of H. Let d_N be a new unary predicate and let L_N be the language of Hilbert spaces together with d_N. We denote the L_N structures by $(\mathcal{H}, d_N)$, where $d_N\colon \mathcal{H} \to [0, 1]$ and we want to consider the structures where d_N is a distance to a subset of H. Instead of measuring the actual distance to the subset, we truncate the distance at one. As before, we need to characterize the functions d_N corresponding to distances.

4.1. **The basic theory** T_0**.** We denote by T_0 the theory of Hilbert spaces together with the next two axioms (compare with Theorem 9.11 in [4]):

(1) $\sup_x \min\{1 \dot- d_N(x), \inf_y \max\{|d_N(x) - \|x - y\||, d_N(y)\}\} = 0$
(2) $\sup_x \sup_y [d_N(y) - \|x - y\| - d_N(x)] \leq 0$

We say a point is *black* if $d_N(x) = 0$ and *white* if $d_N(x) = 1$. All other points are gray, darker if $d(x)$ is close to zero and whiter if $d_N(x)$ is close to one. This terminology follows [10]. From the second axiom we get that d_N is uniformly continuous (with modulus of uniform continuity $\Delta(x, y) = \|x - y\|$). Thus we can apply the tools of continuous model theory to analyze these structures.

4.1. **Lemma.** *Let $(\mathcal{H}, d) \models T_0$ be $\aleph_0$-saturated and let $N = \{x \in H : d_N(x) = 0\}$. Then for any $x \in H$, $d_N(x) = \mathrm{dist}(x, N)$.*

Proof. Let $v \in H$ and let $w \in N$. Then by the second axiom $d_N(v) \leq \|v - w\|$ and thus $d_N(v) \leq \mathrm{dist}(v, N)$.

Now let $v \in H$. If $d_N(v) = 1$ there is nothing to prove, so we may assume that $d_N(v) < 1$. Consider now the set of statements $p(x)$ given by $d_N(x) = 0$, $\|x - v\| = d_N(v)$.

Claim The type $p(x)$ is approximately satisfiable.

Let $\varepsilon > 0$. We want to show that there is a realization of the statements $d_N(x) \leq \varepsilon$, $d_N(v) \leq \|x - v\| + \varepsilon$. By the first axiom there is w such that $d_N(w) \leq \varepsilon$ and $d_N(v) \leq \|v - w\| + \varepsilon$.

Since $(\mathcal{H}, d)$ is $\aleph_0$-saturated, there is $w \in N$ such that $\|v - w\| = d_N(v)$ as we wanted. □

There are several ages that need to be considered. We fix $r \in [0, 1)$ and we consider the class $\mathcal{K}_r$ of all models of T_0 such that $d_N(0) = r$. Note that in all finite dimensional spaces in $\mathcal{K}_r$ we have at least a point v with $d_N(v) = 0$.

4.2. **Notation.** If $(\mathcal{H}_i, d_N^i) \models T_0$ for $i \in \{0, 1\}$, we write $(H_0, d_N^0) \subset (H_1, d_N^1)$ if $H_0 \subset H_1$ and $d_N^0 = d_N^1 \restriction_{H_0}$ (for each sort).

We will work in $\mathcal{K}_r$. We start with free amalgamations:

4.3. **Lemma.** *Let $(\mathcal{H}_0, d_N^0) \subset (\mathcal{H}_i, d_N^i)$ where $i = 1, 2$ and $H_1 \perp_{H_0} H_2$ be Hilbert spaces with distance functions, all of them in $\mathcal{K}_r$. Let $H_3 =$*

$\mathrm{span}\{H_1, H_2\}$ *and let*

$$d_N^3(v) = \min\Big\{\sqrt{d_N^1(P_{H_1}(v))^2 + \|P_{H_2\cap H_0^\perp}(v)\|^2},\\ \sqrt{d_N^2(P_{H_2}(v))^2 + \|P_{H_1\cap H_0^\perp}(v)\|^2}\Big\}.$$

Then $(\mathcal{H}_i, d_N^i) \subset (\mathcal{H}_3, d_N^3)$ *for* $i = 1, 2$, *and* $(\mathcal{H}_3, d_N^3) \in \mathcal{K}_r$.

Proof. For arbitrary $v \in H_1$, $\sqrt{d_N^1(P_{H_1}(v))^2 + \|P_{H_2\cap H_0^\perp}(v)\|^2} = d_N^1(v)$. Since $(\mathcal{H}_0, d_N^0) \subset (\mathcal{H}_i, d_N^i)$ we also have
$\sqrt{d_N^2(P_{H_2}(v))^2 + \|P_{H_1\cap H_0^\perp}(v)\|^2} = \sqrt{d_N^0(P_{H_0}(v))^2 + \|P_{H_1\cap H_0^\perp}(v)\|^2} \geq d_N^1(v)$. Similarly, for any $v \in H_2$, $d_N^3(v) = d_N^2(v)$.

Therefore $(\mathcal{H}_3, d_N^3) \supset (\mathcal{H}_i, d_N^i)$ for $i \in \{1, 2\}$. Now we have to prove that the function d_N^3 that we defined is indeed a distance function.

Geometrically, $d_N^3(v)$ takes the minimum of the distances of v to the selected black subsets of H_1 and H_2. That is, the random subset of the amalgamation of (H_1, d_N^1) and (H_2, d_N^2) is the union of the two random subsets. It is easy to check that $(\mathcal{H}_3, d_N^3) \models T_0$. Since each of (H_1, d_N^1), (H_2, d_N^2) belongs to $\mathcal{K}_r$, we have $d_N^1(0) = r = d_N^2(0)$ and thus $d_N^3(0) = r$. □

The class $\mathcal{K}_0$ also has the JEP: let $(\mathcal{H}_1, d_N^1), (\mathcal{H}_2, d_N^2)$ belong to $\mathcal{K}_0$ and assume that $\mathcal{H}_1 \perp \mathcal{H}_2$. Then $\mathcal{H}_3 = \mathrm{span}(\mathcal{H}_1 \cup \mathcal{H}_2)$ witnesses of the JEP in $\mathcal{K}_0$.

4.4. **Lemma.** *There is a model* $(\mathcal{H}, d_N) \models T_0$ *such that* $\mathcal{H}$ *is a* $2n$*-dimensional Hilbert space and there are orthonormal vectors* $v_1, \ldots, v_n \in H$, $u_1, \ldots, u_n \in H$ *such that* $d_N((u_i + v_j)/2) = \sqrt{2}/2$ *for* $i \leq j$, $d_N(0) = 0$ *and* $d_N((u_i + v_j)/2) = 0$ *for* $i > j$.

Proof. Let H be a Hilbert space of dimension $2n$, and fix some orthonormal basis $\langle v_1, \ldots, v_n, u_1, \ldots, u_n\rangle$ for H. Let $N = \{(u_i+v_j)/2 : i > j\} \cup \{0\}$ and let $d_N(x) = \mathrm{dist}(x, N)$. Then $d_N(0) = 0$ and $d_N((u_i+v_j)/2) = 0$ for $i > j$. Since $\|(u_i + v_j)/2 - (u_k + v_j)/2\| = \sqrt{2}/2$ for $i \neq k$ and $\|(u_i + v_j)/2 - 0\| = \sqrt{2}/2$, we get that $d_N(u_i + v_j) = \sqrt{2}/2$ for $i \leq j$ □

A similar construction can be made in order to get the Lemma with $d_N(0) = r$ for any $r \in [0,1]$. In particular, if we fix an infinite cardinal κ and we amalgamate all possible pairs (H,d) in $\mathcal{K}_r$ for $\dim(H) \leq \kappa$, the theory of the resulting structure will be unstable.

4.2. **The model companion.**

4.2.1. *Basic notations.* We now provide the axioms of the model companion of $T_0 \cup \{d_N(0) = 0\}$.

Call T_{d0} the theory of the structure built out of amalgamating all separable Hilbert spaces together with a distance function belonging to the age $\mathcal{K}_0$. Informally speaking, $T_{d0} = Th(\varinjlim(\mathcal{K}_0))$. We show how to axiomatize T_{d0}.

The idea for the axiomatization of this part (unlike our third example, in next section) follows the lines of Theorem 2.4 of [8]. There are however important differences in the behavior of algebraic closures and independence, due to the metric character of our examples.

Let (M, d_N) in $\mathcal{K}_0$ be an existentially closed structure and take some extension $(M_1, d_N) \supset (M, d_N)$. Let $\bar{x} = (x_1, \ldots, x_{n+k})$ be elements in $M_1 \setminus M$ and let $z_1, \ldots, z_{n+k}$ be their projections on M. Assume that for $i \leq n$ there are $\bar{y} = (y_1, \ldots, y_n)$ in $M_1 \setminus M$ that satisfy $d_N(x_i) = \|x_i - y_i\|$ and $d_N(y_i) = 0$. Also assume that for $i > n$, the witnesses for the distances to the black points belong to M, that is, $d_N^2(x_i) = \|x_i - z_i\|^2 + d_N^2(z_i)$ for $i > n$. Also, let us assume that all points in $\bar{x}, \bar{y}$ live in a ball of radius L around the origin. Let $\bar{u} = (u_1, \ldots, u_n)$ be the projection of $\bar{y} = (y_1, \ldots, y_n)$ over M.

Let $\varphi(\bar{x}, \bar{y}, \bar{z}, \bar{u})$ be a formula such that $\varphi(\bar{x}, \bar{y}, \bar{z}, \bar{u}) = 0$ describes the values of the inner products between all the elements of the tuples, that is, it determines the (Hilbert space) geometric locus of the tuple $(\bar{x}, \bar{y}, \bar{z}, \bar{u})$. The statement $\varphi(\bar{x}, \bar{y}, \bar{z}, \bar{u}) = 0$ expresses the position of the potentially new points $\bar{x}$, $\bar{y}$ with respect to their projections into a model. Since $d_N(x_i) =$

$\|x_i - y_i\|$ and $d_N(y_i) = 0$, we have $\|x_i - y_j\| \geq \|x_i - y_i\|$ for $j \leq n$, $i \leq n$. Also, for $i > n$, $d_N^2(x_i) = \|x_i - z_i\|^2 + d_N^2(z_i)$, and get $\|x_i - y_j\|^2 \geq \|x_i - z_i\|^2 + d_N^2(z_i)$ for $j \leq n$.

Note that as $(M_1, d_N) \supset (M, d_N)$, for all $z \in M$, $d_N^2(z) \leq \|z - y_i\|^2 = \|z - u_i\|^2 + \|y_i - u_i\|^2$ for $i \leq n$. We may also assume that there is a positive real η_φ such that $\|x_i - z_i\| \geq \eta_\varphi$ for $i \leq n + k$ and $\|y_i - u_i\| \geq \eta_\varphi$ for $i \leq n$.

4.2.2. *An informal description of the axioms.* We want to express that for any parameters $\bar{z}$, $\bar{u}$ in the structure
if we can find realizations $\bar{x}$, $\bar{y}$ of $\varphi(\bar{x}, \bar{y}, \bar{z}, \bar{u}) = 0$ such that for all w and $i \leq n$, $d_N^2(w) \leq \|w - u_i\|^2 + \|u_i - y_i\|^2$, $\|x_i - y_i\|^2 \leq \|x_i - z_i\|^2 + d_N^2(z_i)$ for $i \leq n$, $\|x_i - y_j\|^2 \geq \|x_i - z_j\|^2 + d_N^2(z_j)$ for $i > n$ and $j \leq n$,
then there are tuples $\bar{x}'$, $\bar{y}'$ such that $\varphi(\bar{x}', \bar{y}', \bar{z}, \bar{u}) = 0$, $d_N(y_i') = 0$, $d_N(x_i') = \|x_i' - y_i'\|$ for $i \leq n$ and $d_N^2(x_j) = \|x_j - z_j\|^2 + d_N^2(z_j)$ for $j > n$.

That is, for any $\bar{z}$, $\bar{u}$ in the structure, if we can find realizations $\bar{x}$, $\bar{y}$ of the Hilbert space locus given by φ, and we prescribe "distances" d_N that do not clash with the d_N information we already had, in such a way that for $i \leq n$, the y_i's are black and are witnesses for the distance to the black set for the x_i's, and for $i > n$ the x_i's do not require new witnesses, then we can actually find arbitrarily close realizations, *with the prescribed distances.*

The only problem with this idea is that we do not have an implication in continuous logic. We replace the expression "$p \to q$" by a sequence of approximations indexed by ε.

4.2.3. *The axioms of T_N.*

4.5. **Notation.** Let $\bar{z}, \bar{u}$ be tuples in M and let $x \in M_1$. By $P_{\text{span}(\bar{z}\bar{u})}(x)$ we mean the projection of x in the space spanned by $(\bar{z}, \bar{u})$.

For fixed $\varepsilon \in (0, 1)$, let $f\colon [0, 1] \to [0, 1]$ be a continuous function such that whenever $\varphi(\bar{t}) < f(\varepsilon)$ and $\varphi(\bar{t}') = 0$, then

(a): $\|P_{\text{span}(\bar{z}\bar{u})}(x_i) - z_i\| < \varepsilon$.

(b): $\|P_{\mathrm{span}(\bar z\bar u)}(y_i)-u_i\|<\varepsilon$.

(c): $|\|t_i-t_j\|-\|t'_i-t'_j\||<\varepsilon$ where $\bar t$ is the concatenation of $\bar x,\bar y,\bar z,\bar u$.

Choosing ε small enough, we may assume that

(d): $\|x_i-P_{\mathrm{span}(\bar z\bar u)}(x_i)\|\geq \eta_\varphi/2$ for $i\leq n+k$.

(e): $\|y_i-P_{\mathrm{span}(\bar z\bar u)}(y_i)\|\geq \eta_\varphi/2$ for $i\leq n$.

Let $\delta=2\sqrt{\varepsilon(L+2)}$ and consider the following axiom $\psi_{\varphi,\varepsilon}$ (which we write as a positive bounded formula for clarity) where the quantifiers range over a ball of radius $L+1$:

$$\forall\bar z\forall\bar u\Big(\forall\bar x\forall\bar y\varphi(\bar x,\bar y,\bar z,\bar u)\geq f(\varepsilon)\vee\exists w\bigvee_{i\leq n}(d_N^2(w)\geq\|w-u_i\|^2+\|y_i-u_i\|^2+\varepsilon^2)\vee\bigvee_{i>n,j\leq n}(\|x_i-y_j\|^2\leq\|x_i-z_i\|^2+d_N^2(z_i)+\varepsilon^2)\vee\bigvee_{i,j\leq n,j\neq i}(\|x_i-y_j\|\leq\|x_i-y_i\|-\varepsilon)\vee\bigvee_{i\leq n}(\|x_i-z_i\|^2+d_N^2(z_i)\leq\|x_i-y_i\|^2-\varepsilon^2)$$

$$\vee$$

$$\vee\exists\bar x\exists\bar y\big[(\varphi(\bar x,\bar y,\bar z,\bar u)\leq f(\varepsilon)\wedge\bigwedge_{i\leq n}d_N(y_i)\leq\delta)\wedge\bigwedge_{i\leq n}|d_N(x_i)-\|x_i-y_i\||\leq 2\delta)\wedge\bigwedge_{i>n}|d_N^2(x_i)-\|x_i-z_i\|^2-d_N^2(z_i)|\leq 4\delta L\big]\Big)$$

Let T_N be the theory T_0 together with this scheme of axioms $\psi_{\varphi,\varepsilon}$ indexed by all Hilbert space geometric locus formulas $\varphi(\bar x,\bar y,\bar z,\bar u)=0$ and $\varepsilon\in(0,1)\cap\mathbb{Q}$. The radius of the ball that contains all elements, L, as well as n and k are determined from the configuration of points described by the formula $\varphi(\bar x,\bar y,\bar z,\bar u)=0$.

4.2.4. *Existentially closed models of T_0.*

4.6. **Theorem.** *Assume that $(M,d_N)\models T_0$ is existentially closed. Then $(M,d_N)\models T_N$.*

Proof. Fix $\varepsilon>0$ and φ as above. Let $\bar z\in M^{n+k}$, $\bar u\in M^n$ and assume that there are $\bar x,\bar y$ with $\varphi(\bar x,\bar y,\bar z,\bar u)<f(\varepsilon)$ and $d_N^2(w)<\|w-u_i\|^2+\|y_i-u_i\|^2+\varepsilon^2$ for all $w\in M$, $\|x_i-y_i\|^2<\|x_i-z_i\|^2+d_N^2(z_i)+\varepsilon^2$ for $i\leq n$, $\|x_i-y_j\|>\|x_i-y_i\|-\varepsilon$ for $i,j\leq n$, $i\neq j$, $\|x_i-y_j\|^2>\|x_i-z_i\|^2+d_N^2(z_j)+\varepsilon^2$ for $i>n$, $j\leq n$. Let $\varepsilon'<\varepsilon$ be such that $\varphi(\bar x,\bar y,\bar z,\bar u)<f(\varepsilon')$ and

(f): $d_N^2(w)<\|w-u_i\|^2+\|y_i-u_i\|^2+\varepsilon'^2$ for all $w\in M$.

(g1): $\|x_i - y_i\|^2 > \|x_i - z_i\|^2 + d_N(z_i) + \varepsilon'^2$ for $i \leq n$.

(g2): $\|x_i - y_j\| > \|x_i - y_i\| - \varepsilon'$ for $i, j \leq n$, $i \neq j$

(h): $\|x_i - y_j\|^2 \geq \|x_i - z_i\|^2 + d_N^2(z_i) + \varepsilon'^2$ for $i > n$, $j \leq n$.

We construct an extension $(H, d_N) \supset (M, d_N)$ where the conclusion of the axiom indexed by ε' holds. Since (M, d_N) is existentially closed and the conclusion of the axiom is true for (H, d_N) replacing ε for $\varepsilon' < \varepsilon$, then the conclusion of the axiom indexed by ε will hold for (M, d_N).

So let $H \supset M$ be such that $\dim(H \cap M^\perp) = \infty$. Let $a_1, \dots, a_{n+k}$ and $c_1, \dots, c_n \in H$ be such that $\operatorname{tp}(\bar{a}, \bar{c}/\bar{z}\bar{u}) = \operatorname{tp}(\bar{x}, \bar{y}/\bar{z}\bar{u})$ and $\bar{a}\bar{c} \mathop{\smile\!\!\!\!\!\!\,|}_{\bar{z}\bar{u}} M$. We can write $a_i = a_i' + z_i'$ and $c_i = c_i' + u_i'$ for some $z_i', u_i' \in M$ and $a_i', c_i' \in M^\perp$. By (d) and (e) $\|a_i'\| \geq \eta/2$ for $i \leq n + k$ and $\|c_i'\| \geq \eta/2$ for $i \leq n$. Now let $\hat{c}_i = c_i' + u_i' + \delta' c_i'/\|c_i'\|$, where $\delta' = \sqrt{2\varepsilon'(L+2)}$.

Let the black points in H be the ones from M plus the points $\hat{c}_1, \dots, \hat{c}_n$. Now we check that the conclusion of the axiom $\psi_{\varphi,\varepsilon'}$ holds.

(1) $\varphi(\bar{a}, \bar{c}, \bar{z}, \bar{u}) \leq f(\varepsilon')$ since $\operatorname{tp}(\bar{a}, \bar{c}/\bar{z}\bar{u}) = \operatorname{tp}(\bar{x}, \bar{y}/\bar{z}\bar{u})$.

(2) Since $\|c_i - \hat{c}_i\| \leq \delta'$ and $\hat{c}_i$ is black we have $d_N(c_i) \leq \delta'$.

(3) We check that the distance from a_i to the black set is as prescribed for $i \leq n$. $d_N(a_i) \leq \|a_i - \hat{c}_i\| \leq \|a_i - c_i\| + \delta'$ for $i \leq n$.

Also, for $i \neq j, i, j \leq n$, using $(g2)$ we prove $\|a_i - \hat{c}_j\| \geq \|a_i - c_j\| - \delta' \geq \|a_i - c_i\| - \varepsilon' - \delta' \geq \|a_i - c_i\| - 2\delta'$. Finally by (a) $\|a_i - P_M(a_i)\|^2 + d_N^2(P_M(a_i)) \geq (\|a_i - z_i\| - \varepsilon')^2 + (d_N(z_i) - \varepsilon')^2 \geq \|a_i - z_i\|^2 - 2L\varepsilon' + \varepsilon'^2 + d_N^2(z_i) - 2\varepsilon' + \varepsilon'^2$ and by $(g1)$, we get $\|a_i - z_i\|^2 - 2L\varepsilon' + \varepsilon'^2 + d_N^2(z_i) - 2\varepsilon' + \varepsilon'^2 \geq \|a_i - c_i\|^2 - 2L\varepsilon' - 2\varepsilon' \geq \|a_i - c_i\|^2 - 4\delta'^2$.

(4) We check that $d_N(a_i)$ is as desired for $i > n$. Clearly $\|a_j - \hat{c}_i\| \geq \|a_j - c_i\| - \delta'$, so $\|a_j - \hat{c}_i\|^2 \geq \|a_j - c_i\|^2 + \delta'^2 - 2\delta' 2L$ and by (h) we get $\|a_j - c_i\|^2 + \delta'^2 - 4\delta' L \geq \|a_j - z_j\|^2 + d_N^2(z_j) - 4\delta' L - \varepsilon'^2 + \delta'^2 \geq \|a_j - z_j\|^2 + d_N^2(z_j) - 4\delta' L$.

It remains to show that $(M, d_N) \subset (H, d_N)$, i.e., the function d_N on H extends the function d_N on M . Since we added the black points in the ball of radius $L+1$, we only have to check that for any $w \in M$ in the ball of radius $L+2$, $d_N^2(w) \leq \|w-\hat{c}_i\|^2 = \|w-u_i'\|^2 + \|c_i' + \delta'(c_i'/\|c_i'\|)\|^2$.

But by (f) $d_N^2(w) \leq \|w-u_i\|^2 + \|c_i-u_i\|^2 + \varepsilon'^2$, so it suffices to show that

$$\|w-u_i\|^2 + \|c_i-u_i\|^2 + \varepsilon'^2 \leq \|w-u_i'\|^2 + \|c_i'\|^2 + 2\delta'\|c_i'\| + \delta'^2$$

By (a) $\|w-u_i'\|^2 \geq (\|w-u_i\| - \varepsilon')^2$ and is enough to prove that

$$\|w-u_i\|^2 + \|c_i-u_i\|^2 + \varepsilon'^2 \leq (\|w-u_i\| - \varepsilon')^2 + \|c_i'\|^2 + 2\delta'\|c_i'\| + \delta'^2$$

But $(\|w-u_i\|-\varepsilon')^2 + \|c_i'\|^2 + 2\delta'\|c_i'\| + \delta'^2 = \|w-u_i\|^2 - 2\varepsilon'\|w-u_i\| + \varepsilon'^2 + \|c_i'\|^2 + 2\delta'\|c_i'\| + \delta'^2$ and $\|c_i-u_i\|^2 \leq \|c_i-u_i'\|^2 + 2\varepsilon'\|c_i-u_i'\| + \varepsilon'^2 = \|c_i'\|^2 + 2\varepsilon'\|c_i'\| + \varepsilon'^2$. Thus, after simplifying, we only need to check $2\varepsilon'\|w-u_i\| + \varepsilon'^2 \leq \delta'^2$ which is true since $2\varepsilon'\|w-u_i\| + \varepsilon'^2 \leq 2\varepsilon'(2L+2) + \varepsilon'^2 \leq 4\varepsilon'(L+2)$. □

4.7. **Theorem.** *Assume that $(M, d_N) \models T_N$. Then (M, d_N) is existentially closed.*

Proof. Let $(H, d_N) \supset (M, d_N)$ and assume that (H, d_N) is $\aleph_0$-saturated. Let $\psi(\bar{x}, \bar{v})$ be a quantifier free L_N-formula, where $\bar{x} = (x_1, \ldots x_{n+k})$ and $\bar{v} = (v_1, \ldots v_l)$. Suppose that there are $a_1, \ldots, a_{n+k} \in H \setminus M$ and $e_1, \ldots e_l \in M$ such that $(H, d_N) \models \psi(\bar{a}, \bar{e}) = 0$. After enlarging the formula ψ if necessary, we may assume that $\psi(\bar{x}, \bar{v}) = 0$ describes the values of $d_N(x_i)$ for $i \leq n+k$, the values of $d_N(v_j)$ for $j \leq l$ and the inner products between those elements. We may assume that for $i \leq n$ there is $\rho > 0$ such that $d_N(a_i) - d(a_i, z) \geq 2\rho$ for all $z \in M$ with $d_N(z) \leq \rho$. Since (H, d_N) is $\aleph_0$-saturated, there are $c_1, \ldots c_n \in H$ such that $d_N(a_i) = \|a_i - c_i\|$ and $d_N(c_i) = 0$. Then $d(c_i, M) \geq \rho$. Fix $\varepsilon > 0$, $\varepsilon < \rho, 1$. We may also assume that for $i > n$, $|d_N^2(a_i) - \|a_i - P_M(a_i)\|^2 - d_N^2(P_M(a_i))| \leq \varepsilon/2$. Also, assume that all points mentioned so far live in a ball of radius L around the origin. Let $b_1, \ldots, b_{n+k} \in M$ be the projections of $a_1, \ldots, a_{n+k}$ onto M and let $d_1, \ldots, d_n \in M$ be the projections of $c_1, \ldots, c_n$ onto M. Let $\varphi(\bar{x}, \bar{y}, \bar{z}, \bar{u}) = 0$ be an L-statement that describes the inner products between the elements listed and such that $\varphi(\bar{a}, \bar{c}, \bar{b}, \bar{d}) = 0$. Using the axioms we can find $\bar{a}'$, $\bar{c}'$ in M such that $\varphi(\bar{a}', \bar{c}', \bar{b}, \bar{d}) \leq f(\varepsilon)$, $d_N(c_i') \leq \delta$ for $i \leq n$, $|d_N(a_i') - \|a_i' - c_i'\|| \leq \delta$ for $i \leq n$ and $|d_N^2(a_i) - \|a_i - b_i\|^2 - d_N^2(b_i)| \leq 4L\delta$, where $\delta = \sqrt{2\varepsilon(L+2)}$. Since $\varepsilon > 0$ was arbitrary we get $(M, d_n) \models \inf_{x_1} \dots \inf_{x_{n+k}} \psi(\bar{x}, \bar{v}) = 0$. $\square$

4.2.5. *Model theoretic analysis of T_N.* For the rest of this section we will study the theory T_N.

We write tp for types of elements in the language L and tp_N for types of elements in the language L_N. Similarly we denote by acl_N the algebraic closure in the language L_N and by acl the algebraic closure for pure Hilbert spaces. Recall that for a set A, $\text{acl}(A) = \text{dcl}(A)$, and this corresponds to the closure of the space spanned by A (Fact 2.2).

4.8. **Observation.** The theory T_N does not have elimination of quantifiers. We use the characterization of quantifier elimination given in Theorem 8.4.1 from [9]. Let H_1 be a two dimensional Hilbert space, let $\{u_1, u_2\}$ be an orthonormal basis for H_1 and let $N_1 = \{0, u_0 + \frac{1}{4}u_1\}$ and let $d_N^1(x) = \min\{1, \text{dist}(x, N_1)\}$. Then $(H_1, d_N^1) \models T_0$. Let $a = u_0$, $b = u_0 - \frac{1}{4}u_1$ and $c = u_0 + \frac{1}{4}u_1$. Note that $d_N^1(b) = \frac{1}{2}$. Let $(H_1', d_N^1) \supset (H_1, d_N^1)$ be existentially closed. Now let H_2 be an infinite dimensional separable Hilbert space and let $\{v_i : i \in \omega\}$ be an orthonormal basis. Let $N_2 = \{x \in H : \|x - v_1\| = \frac{1}{4}, P_{\text{span}(v_1)}(x) = v_1\} \cup \{0\}$ and let $d_N^2(x) = \min\{1, \text{dist}(x, N_2)\}$. Let $(H_2', d_N^2) \supset (H_2, d_N^2)$ be existentially closed. Then $(\text{span}(a), d_N^1 \restriction_{\text{span}(a)}) \cong (\text{span}(v_1), d_N^2 \restriction_{\text{span}(v_1)})$ and they can be identified say by a function F. But (H_1', d_N^1) and (H_2', d_N^2) cannot be amalgamated over this common substructure: If they could, then we would have $\text{dist}(F(b), v_1 + \frac{1}{4}v_i) = \text{dist}(b, v_1 + \frac{1}{4}v_i) < \frac{1}{2}$ for some $i > 1$ and thus $d_N^1(b) < \frac{1}{2}$, a contradiction.

In this case, the main reason for this failure of amalgamation resides in the fact that $(\text{span}(a), d_N^1 \restriction_{\text{span}(a)}) \cong (\text{span}(v_1), d_N^2 \restriction_{\text{span}(v_1)})$ is not a model of T_0: informally, the distance values around v_1 are “deflected” by an “external attractor" (the black point $u_0 + \frac{1}{4}u_1$ or the black ring orthogonal to v_1 at distance $\frac{1}{4}$) that the subspace $(\text{span}(a), d_N^1 \restriction_{\text{span}(a)})$ simply cannot see. This violates Axiom (1) in the description of T_0. This "noise external to the substructure" accounts for the failure of amalgamation, and ultimately for the lack of quantifier elimination.

In [8, Corollary 2.6], the authors show that the algebraic closure of the expansion of a simple structure with a generic subset corresponds to the algebraic in the original language. However, in our setting, the new algebraic closure $\text{acl}_N(X)$ does not agree with the old algebraic closure $\text{acl}(X)$:

4.9. **Observation.** The previous construction shows that acl_N does not coincide with acl. Indeed, $c \in \text{acl}_N(a) \setminus \text{acl}(a)$ - the set of solutions of the type $\text{tp}_N(c/a)$ is $\{c\}$, but $c \notin dcl(a)$ as $c \notin \text{span}(a)$.

However, models of the basic theory T_0 are L_N-algebraically closed. The proof is similar to [8, Proposition 2.6(3)]:

4.10. **Lemma.** *Let $(M, d_N) \models T_N$ and let $A \subset M$ be such that $A = \text{dcl}(A)$ and $(A, d_N \restriction_A) \models T_0$. Let $a \in M$. Then $a \in \text{acl}_N(A)$ if and only if $a \in A$.*

Proof. Assume $a \notin A$. We will show that $a \notin \text{acl}_N(A)$. Let $a' \models \text{tp}(a/A)$ be such that $a' \perp_A M$. Let (M', d_N) be an isomorphic copy of (M, d_N) over A through $f : M \to_A M'$ such that $f(a) = a'$. We may assume that $M' \perp_A M$. Since $(A, d_N \restriction_A)$ is an amalgamation base, $(N, d_N) = (M \oplus_A M', d_N) \models T_0$. Let $(N', d_N) \supset (N, d_N)$ be an existentially closed structure. Then $\text{tp}_N(a/A) = \text{tp}_N(a'/A)$ and therefore $a \notin \text{acl}_N(A)$. □

As T_N is model complete, the types in the extended language are determined by the existential formulas within them, i.e. formulas of the form $\inf_{\bar{y}} \varphi(\bar{y}, \bar{x}) = 0$

Another difference with the work of Chatzidakis and Pillay is that the analogue to [8, Proposition 2.5] no longer holds. Let a, b, c be as in Observation 4.9; notice that $(\text{span}(a), d_N \restriction_{\text{span}(a)}) \cong (\text{span}(v_1), d_N \restriction_{\text{span}(v_1)})$.

However, $(H_1', d_N, a) \not\equiv (H_2', d_N, v_1)$. Instead, we can show the following weaker version of the Proposition.

4.11. **Proposition.** *Let (M, d_N) and (N, d_N) be models of T_N and let A be a common subset of M and N such that* $(\mathrm{span}(A), d_N \restriction_{\mathrm{span}(A)}) \models T_0$. *Then*

$$(M, d_N) \equiv_A (N, d_N).$$

Proof. Assume that $M \cap N = \mathrm{span}(A)$. Since $(\mathrm{span}(A), d_N \restriction_{\mathrm{span}(A)}) \models T_0$, it is an amalgamation base and therefore we may consider the free amalgam $(M \oplus_{\mathrm{span}(A)} N, d_N)$ of (M, d_N) and (N, d_N) over $(\mathrm{span}(A), d_N \restriction_{\mathrm{span}(A)})$. Let now (E, d_N) be a model of T_N extending $(M \oplus_{\mathrm{span}(A)} N, d_N)$. By the model completeness of T_N, we have that $(M, d_N) \prec (E, d_N)$ and $(N, d_N) \prec (E, d_N)$ and thus $(M, d_N) \equiv_A (N, d_N)$. □

4.3. **Questions around Simplicity.** In this section we define an abstract notion of independence and study its properties.

Fix $(\mathcal{U}, d_N) \models T_N$ be a κ-universal domain.

4.12. **Definition.** Let $A, B, C \subset \mathcal{U}$ be small sets. We say that A is $*$-independent from B over C and write $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$ if $\mathrm{acl}_N(A \cup C)$ is independent (in the sense of Hilbert spaces) from $\mathrm{acl}_N(C \cup B)$ over $\mathrm{acl}_N(C)$. That is, $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$ if for all $a \in \mathrm{acl}_N(A \cup C)$, $P_{\overline{B \cup C}}(a) = P_{\overline{C}}(a)$, where $\overline{B \cup C} = \mathrm{acl}_N(C \cup B)$ and $\overline{C} = \mathrm{acl}_N(C)$.

4.13. **Proposition.** *The relation $\mathop{\smile\!\!\!\!\!\mid}^*$ satisfies the following properties (here A, B, etc., are any small subsets of $\mathcal{U}$):*

(1) *Invariance under automorphisms of $\mathcal{U}$.*
(2) *Symmetry: $A \mathop{\smile\!\!\!\!\!\mid}^*_C B \iff B \mathop{\smile\!\!\!\!\!\mid}^*_C A$.*
(3) *Transitivity: $A \mathop{\smile\!\!\!\!\!\mid}^*_C BD$ if and only if $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$ and $A \mathop{\smile\!\!\!\!\!\mid}^*_{BC} D$.*
(4) *Finite Character: $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$ if and only $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^*_C B$ for all $\bar{a} \in A$ finite.*
(5) *Local Character: If $\bar{a}$ is any finite tuple, then there is countable $B_0 \subseteq B$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^*_{B_0} B$.*

(6) *Extension property over models of* T_0*. If* $(C, d_N \restriction_C) \models T_0$*, then we can find* A' *such that* $\mathrm{tp}_N(A/C) = \mathrm{tp}_N(A'/C)$ *and* $A' \mathop{\smile\!\!\!\!\!\mid}^*_C B$*.*

Proof. (1) Is clear.

(2) It follows from the fact that independence in Hilbert spaces satisfies Symmetry.

(3) It follows from the fact that independence in Hilbert spaces satisfies Transitivity.

(4) Clearly $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$ implies that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^*_C B$ for all $\bar{a} \in A$ finite. On the other hand if $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^*_C B$ for all $\bar{a} \in A$ finite, then for a dense subset A_0 of A, $A_0 \mathop{\smile\!\!\!\!\!\mid}^*_C B$ and thus $A \mathop{\smile\!\!\!\!\!\mid}^*_C B$.

(5) Local Character: let $\bar{a}$ be a finite tuple. Since independence in Hilbert spaces satisfies local character, there is $B_1 \subseteq \mathrm{acl}_N(B)$ countable such that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^*_{B_1} B$. Now let $B_0 \subseteq B$ be countable such that $\mathrm{acl}_N(B_0) \supset B_1$. Then $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^*_{B_0} B$.

(6) Let C be such that $(C, d_N \restriction_C) \models T_0$. Let $D \supset A \cup C$ be such that $(D, d_N \restriction_D) \models T_0$ and let $E \supset B \cup C$ be such that $(E, d_N \restriction_E) \models T_0$. Changing D for another set D' with $\mathrm{tp}_N(D'/C) = \mathrm{tp}_N(D/C)$, we may assume that the space generated by $D' \cup E$ is the free amalgamation of D' and E over C. Clearly D', E are algebraically closed and $D' \mathop{\smile\!\!\!\!\!\mid}^*_C B$.

□

4.14. **Question.** *Does* $\mathop{\smile\!\!\!\!\!\mid}^*$ *satisfy the properties:*

(1) *Extension.*

(2) *Independence Theorem*

More generally, we also have:

4.15. **Question.** *Is* T_N *simple?*

Of course, a positive answer to the first question would give a positive answer to the second one. We conjecture that T_N is indeed a simple theory.

However, we have not yet proved the properties mentioned in Question 4.14 or found another argument for the simplicity of T_N.

When we added a predicate d for the distance to a Hilbert substructure, the corresponding age gave rise to a stable structure. This raises the following question:

4.16. **Question.** *How can we characterize a distance function d so that the resulting Fraïssé structure obtained by amalgamation is stable?*

5. Random predicates

In this section we expand a Hilbert space $(H, +, 0, \langle , \rangle)$ by a random 1-Lipschitz predicate R whose values range on the interval $[0, 1]$ (so for all $a, b \in H$, $|R(a) - R(b)| \leq \|a - b\|$). Let T_0 be the theory of Hilbert spaces together with the statements $\sup_x \sup_y |R(x) - R(y)| \dot{-} \|x - y\| = 0$, $\sup_x R(x) \leq 1$, $\inf_x R(x) \geq 0$.

Our goal for this section is to find a model companion for T_0.

5.1. **Extending models of T_0.** The following construction allows us extend models of T_0 by "normalizing" colors via cones around peaks of their values. By "extending the models" we mean extending the predicate R in a model $(\mathcal{H}_0, R)$ to an arbitrary $\mathcal{H}' \supset \mathcal{H}_0$. This is not a trivial task: we need to be able to deal with arbitrary new "colors" R_i for any finite tuple of points in $H' \setminus H_0$ that are not contradictory with the original R, all the while keeping the correct modulus of uniform continuity. This requires "normalizing" the colors by geometric cones around the peaks of their values.

5.1. **Lemma.** *Let $(H_0, +, 0, \langle , \rangle, R) \models T_0$ and let $(H', +, 0, \langle , \rangle) \supseteq (H_0, +, 0, \langle , \rangle)$. Let $v_1, \dots, v_n \in H'$ and let $R_1, \dots, R_n \in [0, 1]$ be such that for all $x \in H_0$, $|R_i - R(x)| \leq \|v_i - x\|$ and $R_i - R_j \leq \|v_i - v_j\|$. Then there exists $R' : H' \to [0, 1]$ such that $(H', +, 0, \langle , \rangle, R') \supset (H_0, +, 0, \langle , \rangle, R)$, $R'(v_i) = R_i$ and $(H', +, 0, \langle , \rangle, R') \models T_0$*

Proof. For each $y \in H'$ and $i \leq n$, let $R_i(y) = R_i \dot{-} \|v_i - y\|$. Define $R_0(y) = \sup_{x \in H_0}(R(x) \dot{-} \|x - y\|)$. Finally, let

$$R'(y) = \max\{R_0(y), R_1(y), \ldots, R_n(y)\}.$$

For any $y \in H_0$, $R_0(y) = \sup_{x \in H_0}(R(x) \dot{-} \|x - y\|) \geq R(y)$. But for any $x \in H_0$, $R(x) - \|x - y\| \leq R(y)$, so $R_0(y) = R(y)$. On the other hand, for any $1 \leq i \leq n$ and $y \in H_0$, $|R_i - R(y)| \leq \|v_i - y\|$, so $R_i(y) \leq R(y)$. Thus, for any $x \in H_0$, $R'(x) = R(x)$.

Since $R_0(v_i) \leq R_i$ and for any $j \leq n$, $R_i - R_j \leq \|v_i - v_j\|$ we also get $R'(v_i) = R_i$.

Thus the function R' extends R and has the value R_i in v_i for $i = 1, \ldots, n$. It only remains to show that it has the correct modulus of uniform continuity.

Claim 1 For all $y_1, y_2 \in H'$ and $i \leq n$, $|R_i(y_1) - R_i(y_2)| \leq \|y_1 - y_2\|$.

We have $|R_i(y_1) - R_i(y_2)| = (R_i \dot{-} \|v_i - y_1\|) - (R_i \dot{-} \|v_i - y_2\|) \leq (\|v_i - y_2\| - \|v_i - y_1\|) \leq \|y_1 - y_2\|$.

Claim 2 For all $y_1, y_2 \in H'$, $|R_0(y_1) - R_0(y_2)| \leq \|y_1 - y_2\|$.

If $R_0(y_1) = R_0(y_2) = 0$ the result is clear. Assume now that $R_0(y_1) = 0$ and that $R_0(y_2) > 0$. Given $\varepsilon > 0$, there is $a_\varepsilon \in H_0$ such that $0 < R_0(y_2) - (R(a_\varepsilon) - \|y_2 - a_\varepsilon\|) < \varepsilon$. Also $R(a_\varepsilon) - \|y_1 - a_\varepsilon\| \leq 0$, so $R_0(y_2) - R_0(y_1) = R_0(y_2) \leq -\|y_2 - a_\varepsilon\| + \|y_1 - a_\varepsilon\| + \varepsilon \leq \|y_2 - y_1\| + \varepsilon$. Since ε was arbitrary, we get the desired result. Finally assume that $R_0(y_1) > 0$ and that $R_0(y_2) > 0$. Given $\varepsilon > 0$, there are $a_\varepsilon, b_\varepsilon \in H_0$ such that $0 < R_0(y_2) - (R(a_\varepsilon) - \|y_2 - a_\varepsilon\|) < \varepsilon$, $0 < R_0(y_1) - (R(b_\varepsilon) - \|y_1 - b_\varepsilon\|) < \varepsilon$ and $R_0(y_1) \geq R(a_\varepsilon) - \|y_1 - \varepsilon\|$. We have $R_0(y_2) - R_0(y_1) \leq R(a_\varepsilon) - \|y_2 - a_\varepsilon\| + \varepsilon - R(a_\varepsilon) + \|y_1 - a_\varepsilon\| \leq -\|y_2 - a_\varepsilon\| + \|y_1 - a_\varepsilon\| + \varepsilon \leq \|y_2 - y_1\| + \varepsilon$. In a similar way we prove $R_0(y_1) - R_0(y_2) \leq \|y_2 - y_1\| + \varepsilon$. Since ε was arbitrary we get the claim.

Finally, from the previous two claims it follows that for all $y_1, y_2 \in H_1$, $|R'(y_1) - R'(y_2)| \leq \|y_1 - y_2\|$. □

5.2. **The model companion.** Let (M,R) be an existentially closed model of T_0. Let $(M_1,R)\supset(M,R)$ and let $\bar a=(a_1,\dots,a_k)$ be elements in $M_1\setminus M$ and let $b_i=P_M(a_i)$ for $i=1,\dots,k$.

Now let $R_1=R(a_1),\dots,R_k=R(a_k)$ and let $\varphi(\bar x,\bar z)$ be a L-formula such that $\varphi(\bar x,\bar z)=0$ describes the values of the inner products between all the elements of the tuples $\bar a$ and $\bar b$ (so $\varphi=0$ describes $\mathrm{tp}(\bar a,\bar b)$). Assume that $\bar a$, $\bar b$ live in a ball of radius L.

5.2.1. *An informal description of the axioms.* As before, we want to express that for any parameters $\bar z$ in the structure, if we can find realizations $\bar x$ of $\varphi(\bar x,\bar z)=0$ such that for all $y_i\in B_1(z_i)$, $i=1,\dots,n$, $|R_i-R(y_i)|\le\sqrt{\|z_i-y_i\|^2+\|z_i-x_i\|^2}$ and $|R_i-R_j|\le\|x_i-x_j\|$, then there is a tuple $\bar x'$, such that $\varphi(\bar x',\bar z)=0$ and $R(x_i')=R_i$. As in the previous section, since implications do not exist in continuous logic, we will replace the main implication by a sequence of approximations.

5.2.2. *The axioms.* Fix $\varepsilon>0$ and let $f\colon[0,1]\to[0,1]$ be a continuous function such that whenever $\varphi(\bar x,\bar z)<f(\varepsilon)$ and $\varphi(\bar x',\bar z')<f(\varepsilon)$, then $|\langle x_i,x_j\rangle-\langle x_i',x_j'\rangle|<\varepsilon$, $|\langle z_i,z_j\rangle-\langle z_i',z_j'\rangle|<\varepsilon$, $|\langle x_i,z_j\rangle-\langle x_i',z_j'\rangle|<\varepsilon$ and for all $y\in B_1(z_i)$, $|(\|y-z_i\|^2+\|z_i-x_i\|^2)-(\|y-P_{\operatorname{span}\bar z}(x_i)\|^2+\|x_i-P_{\operatorname{span}\bar z}(x_i)\|^2)|<\varepsilon$ for all $i,j=1,\dots,k$.

Let T_1 be the theory T_0 together with the following axiom scheme, indexed by $\varepsilon>0$, a configuration $\varphi(\bar x,\bar z)=0$ as above and a k-tuple $\bar R=(R_1,\dots,R_k)$ such that $|R_i-R_j|\le\|x_i-x_j\|$. The quantifiers are taken in a ball of radius $L+1$. Again we write the axioms as positive bounded formulas for clarity.

$\forall z_1\dots\forall z_k\Big(\forall x_1\dots\forall x_k\big([f(\varepsilon)\le\varphi(\bar x,\bar z)\vee$
$\forall y_1\in B_1(z_1)\sqrt{\|x_1-z_1\|^2+\|y_1-z_1\|^2}+\varepsilon\le|R_1-R(y_1)|\vee\dots\vee$
$\forall y_k\in B_1(z_k)\sqrt{\|x_k-z_k\|^2+\|y_k-z_k\|^2}+\varepsilon\le|R_k-R(y_k)|]\vee$
$\exists x_1'\dots,\exists x_k'\exists_{x_1''}\dots,\exists x_k''\varphi(\bar x'',\bar z)\le f(\varepsilon)\wedge|R(x_1')-R_1|=0\wedge\dots\wedge|R(x_k')-$
$R_k|\wedge\|x_1'-x_1''\|\le\sqrt{3\varepsilon+\varepsilon^2}\big)\Big)$

5.2. **Theorem.** *Every existentially closed model of T_0 is a model of T_1.*

Proof. Let $(H,+,0,\langle,\rangle,R) \models T_0$ be existentially closed, let $\varepsilon > 0$ be a real number, let $\varphi(\bar{x},\bar{z}) = 0$ describe a Hilbert space geometric locus of points (i.e. the values of their inner products), let $\bar{R} = (R_1,\dots,R_k) \in [0,1]^k$ be compatible with the configuration (i.e. $|R_i - R_j| \leq \|x_i - x_j\|$) for $i,j \leq k$. We show $(H,+,0,\langle,\rangle,R)$ satisfies the $(\varepsilon,\varphi,\bar{R})$-th axiom of T_1.

Let $(H_1,+,0,\langle,\rangle) \supset (H,+,0,\langle,\rangle)$ such that $\dim(H_1 \cap H^\perp) = 2k$ and let $t_1,\dots,t_k,u_1,\dots,u_k$ be an orthonormal basis for $H_1 \cap H^\perp$. Let $z_1,\dots,z_k \in H$ and assume there are $x_1,\dots,x_k \in H_1$ such that for all $y_1,\dots,y_k \in H$ the following conditions hold:

(1): $\sqrt{\|x_i - z_i\|^2 + \|z_i - y_i\|^2} + \varepsilon > |R_i - R(y_i)|$

(2): $\varphi(\bar{x},\bar{z}) < f(\varepsilon)$

Let now $x_1'',\dots,x_k'' \in \operatorname{span}\{z_1,\dots,z_k,t_1,\dots,t_k\}$ be such that

$$\operatorname{tp}(x_1'',\dots,x_k''/z_1,\dots,z_k) = \operatorname{tp}(x_1,\dots,x_k/z_1,\dots,z_k)$$

and let, for $i = 1,\dots,k$, $x_i' = x_i'' + \sqrt{3\varepsilon + \varepsilon^2}u_i$. Note that

(3): $P_H(x_i'') = P_{\operatorname{span}(\bar{z})}(x_i'')$

We first check that these x_i', x_j'' satisfy the conclusion of the axiom. Clearly, $\varphi(\bar{x}'',\bar{z}) \leq f(\varepsilon)$ and $\|x_i - x_i''\| \leq \sqrt{3\varepsilon + \varepsilon^2}$. The other conclusions deal with the values of R. We will extend R to H_1 by checking that the conditions of Lemma 5.1 hold.

Claim For all $w \in H$ and $i = 1,\dots,k$, $\|w - x_i'\| \geq |R_i - R(w)|$.

Let $w \in H$ and assume that $\|w - x_i'\| \leq 1$ (otherwise $|R(w) - R(x_i')|$ is trivially $\leq \|w - x_i'\|$). Then $\|w - x_i''\|^2 = \|w - P_{\operatorname{span}\bar{z}}(x_i'')\|^2 + \|x_i'' - P_{\operatorname{span}(\bar{z})}(x_i'')\|^2$. By the definition of $f(\varepsilon)$, we have

$$(4): \ |(\|w - z_i\|^2 + \|z_i - x_i''\|^2) - (\|w - P_{\operatorname{span}\bar{z}}(x_i'')\|^2 + \|x_i'' - P_{\operatorname{span}\bar{z}}(x_i'')\|^2)| < \varepsilon.$$

By condition (1), $|R_i - R(w)| < \varepsilon + \sqrt{\|x_i'' - z_i\|^2 + \|z_i - w\|^2}$. Therefore $|R_i - R(w)| < \sqrt{\varepsilon^2 + 2\varepsilon\sqrt{\|x_i'' - z_i\|^2 + \|z_i - w\|^2} + \|x_i'' - z_i\|^2 + \|z_i - w\|^2} \leq$

$\sqrt{\varepsilon^2 + 2\varepsilon\sqrt{\|x_i'' - w\|^2 + \varepsilon} + \|x_i'' - w\|^2 + \varepsilon}$ (the last inequality holds by (3) and (4)). Since $\|w - x_i'\| \leq 1$ then $\|w - x_i'\|^2 \leq 1$ and $\|w - x_i''\|^2 + \|x_i' - x_i''\|^2 \leq 1$. Therefore $\|x_i'' - w\|^2 + \varepsilon < 1$ and $|R_i - R(w)| < \sqrt{\varepsilon^2 + 2\varepsilon + \|x_i'' - w\|^2 + \varepsilon} = \sqrt{\varepsilon^2 + 3\varepsilon + \|x_i'' - w\|^2} = \|x_i' - w\|^2$.

Since the conditions of Lemma 5.1 hold, we can extend R to a function defined on H_1, where $R(x_i') = R_i$ for $i \leq k$. Since $(\mathcal{H}_0, R)$ is existentially closed and $(\mathcal{H}_1, R) \supset (\mathcal{H}_0, R_0)$, the axiom also holds in $(\mathcal{H}_0, R)$. □

5.3. **Theorem.** *Every model of T_1 is an existentially closed model of T_0.*

Proof. Let $(\mathcal{H}, R)$ be a model of T_1. We now show $(\mathcal{H}, R)$ is existentially closed. Let $(\mathcal{H}_1, R) \supset (\mathcal{H}, R)$, $a_1, \dots, a_k \in H_1 \setminus H$ and $c_1, \dots, c_n \in H$. Let $\bar{a} = (a_1, \dots, a_k)$, $\bar{c} = (c_1, \dots, c_n)$. Pick now a quantifier free formula $\psi(x_1, \dots, x_k, y_1, \dots, y_n)$ such that $(\mathcal{H}_1, R) \models \psi(\bar{a}, \bar{c}) = 0$. This formula describes the inner products and the colors of some of the linear combinations of the a_i's and the c_j's. Without loss of generality (just enlarge the tuple of a_i's if needed) we may assume the colors described by ψ are exactly those of the a_i's. Let now $b_1 = P_H(a_1), \dots, b_k = P_H(a_k)$ and let $\varphi(x_1, \dots, x_k, z_1, \dots, z_k) = 0$ describe (as above) the Hilbert space geometric locus of $(\bar{a}, \bar{b})$, that is, the inner products between the elements listed. Let $\varepsilon > 0$ and let $R_i = R(a_i)$ for $i = 1, \dots, k$. Since the $(\varepsilon, \varphi, \bar{R})$-th axiom of T_1 holds in $(\mathcal{H}, R)$, letting $b_1, \dots, b_k$ be instances of $z_1, \dots, z_k$, the axiom guarantees the existence of $a_1', \dots, a_k'$ such that $d(\mathrm{tp}(\bar{a}/\bar{b}), \mathrm{tp}(\bar{a}'/\bar{b})) < \varepsilon$ and $R(a_i') = R(a_i)$. In particular, $(\mathcal{H}_0, R) \models \inf_{x_1} \dots \inf_{x_k} \psi(\bar{x}, \bar{c}) = 0$. □

5.3. **Further model theoretic properties of T_0, $T_{0,0}$, T_1 and $T_{1,0}$.** The theory T_1 is not complete as the value of $R(0)$ may vary (compare with Corollary 2.6, part (1) in [8]). We will study the theory $T_{1,0} = T_1 \cup \{R(0) = 0\}$.

5.4. **Lemma.** *Let $(\mathcal{H}_0, R_0), (\mathcal{H}_1, R_1) \models T_{1,0}$. Then $(\mathcal{H}_0, R_0) \equiv (\mathcal{H}_1, R_1)$.*

Proof. Let $\mathcal{H}_2$ be the Hilbert space generated by the free amalgam of $\mathcal{H}_0$ and $\mathcal{H}_1$ (so $H_2 = H_0 \oplus H_1$). Now let, for $v \in H_2$, $R_2(v) = \max\{0, \sup_{x\in H_1}(R_1(x) - \|v - x\|), \sup_{x\in H_0}(R_0(x) - \|v - x\|)\}$. Then $(\mathcal{H}_2, R_2) \models T_0$ and $(\mathcal{H}_0, R_0) \subset (\mathcal{H}_2, R_2)$, $(\mathcal{H}_1, R_1) \subset (\mathcal{H}_2, R_2)$. Finally let $(\mathcal{H}_3, R_3) \supset (\mathcal{H}_2, R_2)$ be existentially closed. Then $(\mathcal{H}_3, R_3) \equiv (\mathcal{H}_1, R_1)$ and $(\mathcal{H}_3, R_3) \equiv (\mathcal{H}_2, R_2)$. □

The proof above shows that $T_{1,0}$ is complete and has the joint embedding property. Let $T_{0,0} = T_0 \cup \{R(0) = 0\}$. We will now prove that $T_{0,0}$ does not have the amalgamation property:

5.5. **Lemma.** *The theory $T_{0,0}$ does not have the amalgamation property.*

Proof. Let $\mathcal{H}_0$ be a 1-dimensional Hilbert space; set $R(x) = \min\{1/2, \|x\|\}$ for all $x \in H_0$. Clearly $(H_0, R) \models T_{0,0}$. Let $\mathcal{H}_1$ be a 2-dimensional Hilbert space with orthonormal base $\{v_1, v_2\}$ and define $R_1(\alpha v_1 + \beta v_2) = \min\{1, \min\{1/2, \|x\|\} + \|\beta v_2\|\}$. Again $(\mathcal{H}_1, R_1) \models T_{0,0}$ and we can see $(\mathcal{H}_0, R)$ as a substructure of $(\mathcal{H}_1, R_1)$ by identifying it with the space spanned by v_1. Finally let $\mathcal{H}_2$ be a 2-dimensional Hilbert space with orthonormal base $\{w_1, w_2\}$ and define $R_2(\alpha w_1 + \beta w_2) = \max\{0, \min\{1/2, \|x\|\} - \|\beta v_2\|\}$. Again, we can see $(\mathcal{H}_0, R)$ as a substructure of $(\mathcal{H}_2, R_2)$ by identifying it with the space spanned by w_1. Then $R_2((1/2)(w_2 + w_1)) = 0$, $R_2((1/2)(w_1 - w_2)) = 0$, $R_1((1/2)(v_1 + v_2)) = 1$, $R_1((1/2)(v_1 - v_2) = 1$. Assume, in order to get a contradiction that we can amalgamate $(\mathcal{H}_1, R_1)$ and $(\mathcal{H}_2, R_2)$ over $(\mathcal{H}_0, R_0)$. Then identifying v_1 with w_1, we get $\|(1/2)(w_2 + w_1) - (1/2)(v_2 + v_1))\| \leq \sqrt{2}/2$ or $\|(1/2)(w_2 + w_1) + (1/2)(v_2 - v_1)\| \leq \sqrt{2}/2$. Without loss of generality, $\|(1/2)(w_2 + w_1) - (1/2)(v_2 + v_1))\| \leq \sqrt{2}/2$, but then $|R((1/2)(w_2 + w_1) - R((1/2)(v_2 + v_1)| = 1 > \sqrt{2}/2$ a contradiction. □

5.6. **Notation.** We write tp for L-types, tp_R for L_R-types, qftp_R for quantifier free L_R-types. We also write acl, acl_R, dcl, dcl_R for the algebraic closure and the definable closure in the languages L and L_R respectively.

5.7. **Corollary.** *$T_{1,0}$ does not have elimination of quantifiers.*

Proof. The construction above shows that $\mathrm{qftp}_R(v_1) = \mathrm{qftp}_R(w_1)$, whereas $\mathrm{tp}_R(v_1) \neq \mathrm{tp}_R(w_1)$. □

We now give an example that shows that $\mathrm{acl}_R \neq \mathrm{acl}$ (compare with Corollary 2.6, part (3) in [8] and Observation 4.9) in $T_{1,0}$.

5.8. **Example.** Let $\mathcal{H}$ be a two dimensional Hilbert space and let $\{u_0, u_1\}$ be an orthonormal basis for H. Let $v = u_0 + u_1$ and let $R(\alpha u_0 + \beta u_1) = \max\{0, r - \|\alpha u_0 + \beta u_1\|, 1 - \|v - (\alpha u_0 + \beta u_1)\|\}$. Geometrically, R corresponds to a cone of hight one centered at v. For the vector $w = u_0 + 1/2 u_1$, $R(w) = 1/2$. Note that w is the middle point between u_0 and v and that $R(u_0) = 0$. We may assume that $(\mathcal{H}, R) \subset (\mathcal{H}_1, R)$ for some $(\mathcal{H}_1, R) \models T_{1,0}$.

Claim For any $t \in H_1$, if $\|w - t\| = 1/2$ and $R(t) = 1$, then $t = v$.

Otherwise we have $\|u_0 - t\| < 1$ so $R(t) < R(u_0) + 1 = 1$. This proves that the only realization of $\mathrm{tp}(v/w)$ is v, so $v \in \mathrm{dcl}_R(w) \setminus \mathrm{dcl}(w)$.

5.9. **Lemma.** *There is a model $(\mathcal{H}, R) \models T_{0,0}$ such that $\mathcal{H}$ is a $2n+1$-dimensional Hilbert space and there are orthonormal vectors $v_1, \ldots, v_n \in H$, $u_1, \ldots, u_n \in H$ such that $R((u_i + v_j)/2) = \sqrt{2}/2$ for $i \leq j$ and $R((u_i + v_j)/2) = 0$ for $i > j$.*

Proof. Let H be a $2n+1$-dimensional Hilbert space with an orthonormal basis $v_1, \ldots, v_n$, $u_1, \ldots, u_n, w \in H$. Let $N = \{(u_i + v_j)/2 : i > j\} \cup \{0\}$ and let $R(x) = \min\{\mathrm{dist}(x, N), 1\}$. Then $R(0) = 0$ and $R((u_i + v_j)/2) = 0$ for $i > j$. Since $\|(u_i + v_j)/2 - (u_k + v_j)/2\| = \sqrt{2}/2$ for $i \neq k$ and $\|(u_i + v_j)/2 - rw/\|w\|\| = \sqrt{1/2 + r} \geq \sqrt{2}/2$, we get that $R(u_i + v_j) = \sqrt{2}/2$ for $i \leq j$. □

As before, in a saturated model of $T_{1,0}$ we have witnesses of the order property and thus $T_{1,0}$ is not stable.

5.10. **Notation.** Let $(\mathcal{H}, R) \models T_{1,0}$ and let $A \subset H$ be algebraically closed. We say that A is a *free amalgamation base for* $T_{0,0}$ if whenever $(\mathcal{H}_1, R_1) \supset (A, R)$, $(\mathcal{H}_2, R_2) \supset (A, R)$ are models of $T_{0,0}$, there is a model (H_3, R_3) of

$T_{1,0}$ such that $(\mathcal{H}_3, R_3) \supset (A, R)$ and there are isomorphic copies $(\mathcal{H}'_1, R'_1) \supset (A, R)$, $(\mathcal{H}'_2, R'_2) \supset (A, R)$ of $(\mathcal{H}_1, R_1)$, $(\mathcal{H}_2, R_2)$ over (A, R) inside $(\mathcal{H}_3, R_3)$ such that $H'_1 \perp_A H'_2$.

By the proof of Lemma 5.4, $\{0\}$ is a free amalgamation basis for T_0. Models of $T_{1,0}$ are also free amalgamation base for $T_{0,0}$.

5.11. **Lemma.** *Let $(M, d_N) \models T_{1,0}$ and let $A \subset M$ be such that $A = \operatorname{dcl}(A)$ and A is a free amalgamation bases over $T_{0,0}$. Let $a \in M$. Then $a \in \operatorname{acl}_N(A)$ if and only if $a \in A$.*

5.12. **Proposition.** *Let (M, d_N) and (N, d_N) be models of $T_{1,0}$ and let A be a L-definably closed common subset of M and N such that A is a free amalgamation base over $T_{0,0}$. Then*

$$(M, d_N) \equiv_A (N, d_N).$$

The proofs of these two results are similar to Lemma 4.10 and Proposition 4.11 and we leave them to the reader.

5.4. **Questions around Simplicity.** In this subsection we define an abstract notion of independence and study its properties.

Fix $(\mathcal{U}, R) \models T_{1,r}$ be a κ-universal domain.

5.13. **Definition.** Let $A, B, C \subset \mathcal{U}$ be small sets. We say that A is $*$-independent from B over C and write $A \perp^*_C B$ if $\operatorname{acl}_N(A \cup C)$ is independent (in the sense of Hilbert spaces) from $\operatorname{acl}_N(C \cup B)$ over $\operatorname{acl}_N(C)$. That is, $A \perp^*_C B$ if for all $a \in \operatorname{acl}_N(A \cup C)$, $P_{\overline{B \cup C}}(a) = P_{\overline{C}}(a)$, where $\overline{B \cup C} = \operatorname{acl}_N(C \cup B)$ and $\overline{C} = \operatorname{acl}_N(C)$.

5.14. **Proposition.** *The relation $\perp^*$ satisfies the following properties (here A, B, etc., are any small subsets of $\mathcal{U}$):*

(1) *Invariance under automorphisms of $\mathcal{U}$.*
(2) *Symmetry: $A \perp^*_C B \iff B \perp^*_C A$.*

(3) *Transitivity:* $A \downfree^*_C BD$ *if and only if* $A \downfree^*_C B$ *and* $A \downfree^*_{BC} D$.
(4) *Finite Character:* $A \downfree^*_C B$ *if and only* $\bar{a} \downfree^*_C B$ *for all* $\bar{a} \in A$ *finite.*
(5) *Local Character: If* $\bar{a}$ *is any finite tuple, then there is countable* $B_0 \subseteq B$ *such that* $\bar{a} \downfree^*_{B_0} B$.
(6) *Extension property over free amalgamation basis over* T_0. *If the model* $(C, d_N \restriction_C) \models T_0$ *is a free amalgamation basis over* T_0, *then we can find* A' *such that* $\mathrm{tp}_N(A/C) = \mathrm{tp}_N(A'/C)$ *and* $A' \downfree^*_C B$.

The proof is similar to the one presented in the previous section so we leave it to the reader. As before, it is not clear yet how to prove that $\downfree^*$ satisfies Extension and the Independence Theorem.

5.15. **Question.** *Is* T *simple? Does* $\downfree^*$ *satisfy the properties:*

(1) *Extension.*
(2) *Independence Theorem*

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