ORBIFOLD VIRTUAL COHOMOLOGY OF THE SYMMETRIC PRODUCT

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Abstract. The virtual cohomology of an orbifold is a ring structure on the cohomology of the inertia orbifold whose product is defined via the pull-push formalism and the Euler class of the excess intersection bundle. In this paper we calculate the virtual cohomology of a large family of orbifolds, including the symmetric product.

1. Introduction

It was noticed in [LUX07] that the ring structure defined in the homology of the loop space of the symmetric product orbifold (see [LUX]) induces a ring structure on the cohomology of the inertia orbifold, by restricting the structure to the constant loops. This led the authors of [LUX07] to define a ring structure on the inertia orbifold of any orbifold that the authors coined virtual cohomology. This cohomology is defined via the pull-push formalism in as much as the same way that the Chen-Ruan product for orbifolds is defined (see [CR04, FG03]). In [GLS07] the relation between the virtual and the Chen-Ruan cohomology was clarified, namely, that for an almost complex orbifold, its virtual cohomology is isomorphic to the Chen-Ruan cohomology of its cotangent orbifold.

In this paper we give an algorithm to calculate the virtual cohomology of a large family of orbifolds. We first show that for any global quotient orbifold \([Y/G]\), the virtual cohomology \(H^*_\text{virt}(Y, G; \mathbb{Z})\) maps to the group ring \(H^*(Y; \mathbb{Z})[G]\), and therefore, when this map is injective we can see the virtual cohomology as a subring of the group ring. This for example is the case when the inclusions of the fixed point sets \(Y^g \to Y\) induce a monomorphism in homology. We calculate the virtual cohomology of these orbifolds by describing a set of generators in the group ring. In the case of the symmetric product we reduce the set of generators to the lower degree cohomology classes of the fixed point sets of the transpositions.

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2. Virtual Cohomology

Let \( [Y/G] \) be an orbifold with \( Y \) differentiable, compact, oriented and closed, and \( G \) a finite group acting smoothly on \( Y \) preserving the orientation. The inertia orbifold \( I[Y/G] \) is defined as the orbifold

\[
I[Y/G] = \left( \bigsqcup_{g \in G} Y^g \times \{g\} \right)/G
\]

where \( Y^g \) denotes the fixed point set of the element \( g \), we label the components with the elements of the group and \( G \) acts in the following way:

\[
in: \bigsqcup_{g \in G} Y^g \times \{g\} \times G \to \bigsqcup_{g \in G} Y^g \times \{g\}
\]

\[
(x, g, h) \mapsto (xh, h^{-1}gh).
\]

From [LUX07] we know that the virtual intersection product defines a ring structure on the cohomology of the inertia orbifold \( I[Y/G] \), this ring is what the authors in [LUX07] have called virtual cohomology. Let’s recall its definition.

Consider the groups

\[
H^*(Y, G; \mathbb{Z}) := \bigoplus_{g \in G} H^*(Y^g; \mathbb{Z}) \times \{g\}
\]

and for \( g, h \in G \) define the maps

\[
\times: H^*(Y^g; \mathbb{Z}) \times H^*(Y^h; \mathbb{Z}) \to H^*(Y^{gh}; \mathbb{Z})
\]

\[
(\alpha, \beta) \mapsto i_{gh}(i_g^*\alpha \cdot i_h^*\beta \cdot e(Y, Y^g, Y^h))
\]

where \( i_g: Y^g \cap Y^h \to Y^g \), \( i_h: Y^g \cap Y^h \to Y^h \) and \( i_{gh}: Y^g \cap Y^h \to Y^{gh} \) are the inclusion maps, \( e(Y, Y^g, Y^h) \) is the Euler class of the excess bundle of the inclusions \( Y^g \to Y \leftarrow Y^h \) (see [Qui71]) and \( i_{gh} \) is the pushforward map in cohomology.

In [LUX07] it was required that the orbifold be almost complex with a compatible \( G \) action, but for the product to be well defined it is only necessary that the Euler classes of the excess bundles be of even degree. This can be achieved if for all \( g \in G \) the fixed point sets

\[
Y^{g_1 \cdots g_n} := Y^{g_1} \cap \cdots \cap Y^{g_n}
\]

are of even dimension.

The group \( G \) acts on \( H^*(Y, G; \mathbb{Z}) \) in the following way: for \( g, h \in G \) and \( \alpha \in H^*(Y^g; \mathbb{Z}) \) we have

\[
(\alpha, g) \cdot h := ((h^{-1})^*\alpha, h^{-1}gh).
\]

**Definition** (2.2). Let \( [Y/G] \) be an orbifold such that for all \( g_i \in G \) the fixed point sets \( Y^{g_1 \cdots g_n} \) are even dimensional. Then, the group \( H^*(Y, G; \mathbb{Z}) \) together with the ring structure

\[
\cdot: H^*(Y, G; \mathbb{Z}) \times H^*(Y, G; \mathbb{Z}) \to H^*(Y, G; \mathbb{Z})
\]

\[
((\alpha, g), (\beta, h)) \mapsto (\alpha \times \beta, gh)
\]
is what is called the virtual cohomology of the pair \((Y, G)\); we will denote it by \(H^\text{vir}_*(Y, G; \mathbb{Z})\). Moreover, as the ring structure is \(G\)-equivariant with respect to the action of \(G\) on \(H^*(Y, G; \mathbb{Z})\), we define the virtual cohomology of the orbifold \([Y/G]\) as the \(G\) invariant part of the ring \(H^*_\text{vir}(Y, G; \mathbb{R})\), i.e.

\[
H^*_{\text{vir}}([Y/G]; \mathbb{R}) := H^*(Y, G; \mathbb{R})^G.
\]

In what follows we will show how to calculate the virtual cohomology for a large family of orbifolds.

For all \(g \in G\) let \(f_g : Y^g \to Y\) be the inclusion of manifolds and

\[
f_g! : H^*(Y^g; \mathbb{Z}) \to H^*(Y; \mathbb{Z})
\]

be the pushforward in cohomology. Consider the group ring \(H^*(Y; \mathbb{Z})[G]\) of the group \(G\) with coefficients in the ring \(H^*(Y; \mathbb{Z})\), together with the \(G\) action defined by

\[
\left( \sum_i \alpha_i g_i \right) \cdot h := \sum_i ((h^{-1} \cdot \alpha_i) h^{-1} g_i h).
\]

**Theorem (2.3).** The inclusions \(f_g : Y^g \to Y\) induce an equivariant ring homomorphism from the virtual cohomology to the group ring

\[
f : H^*_{\text{vir}}(Y, G; \mathbb{Z}) \to H^*(Y; \mathbb{Z})[G],
\]

\[
(\alpha, g) \mapsto (f_g! \alpha) g.
\]

**Proof.** To show that the map \(f\) is a ring homomorphism we only need to check the commutativity of the following diagram

\[
\begin{array}{ccc}
H^*(Y^g; \mathbb{Z}) \times H^*(Y^h; \mathbb{Z}) & \xrightarrow{f_g! \times f_h!} & H^*(Y; \mathbb{Z}) \times H^*(Y; \mathbb{Z}) \\
\downarrow \times & & \downarrow \times \\
H^*(Y^{gh}; \mathbb{Z}) & \xrightarrow{f_{gh}!} & H^*(Y; \mathbb{Z}).
\end{array}
\]

Consider the diagram of inclusions

\[
\begin{array}{ccc}
Y & \xleftarrow{f_g} & Y^g \\
\downarrow{f_h} & & \downarrow{i_g} \\
Y^h & \xleftarrow{i_h} & Y^{gh}.
\end{array}
\]

It was proven in [LUX07, Lemma 16] by an application of Quillen’s excess intersection formula [Qui71, Prop. 3.3] that for \(\alpha \in H^*(Y^g; \mathbb{Z})\) and \(\beta \in H^*(Y^h; \mathbb{Z})\) one has

\[
f_{gh}! \alpha \cdot f_{gh}! \beta = s(i_g^* \alpha \cdot i_h^* \beta \cdot e(Y, Y^g, Y^h)).
\]

Now, as \(s = f_{gh} \circ i_{gh}\) we have that \(s_i = f_{gh} \circ i_{gh}\) and therefore

\[
f_{gh}! \alpha \cdot f_{gh}! \beta = f_{gh}!(i_{gh}^* i_g^* \alpha \cdot i_h^* \beta \cdot e(Y, Y^g, Y^h)) = f_{gh}!(\alpha \times \beta).
\]
To check that the map $f$ is $G$-equivariant we simply consider the inclusion
\[
\psi: \bigsqcup_{g \in G} Y^g \times \{g\} \to \bigsqcup_{g \in G} Y^g \times \{g\}
\]
\[f_{g,h}: (x, g) \mapsto (f_g x, g).
\]
If we endow the space $\bigsqcup_{g \in G} Y^g \times \{g\}$ with the same $G$-action as in 2.1 then the map $\psi$ becomes $G$-equivariant and we have the commutativity of the following square:
\[
\begin{array}{ccc}
Y^g \times \{g\} & \xrightarrow{f_g} & Y \times \{g\} \\
\downarrow{h} & & \downarrow{h} \\
Y^{gh^{-1}} \times \{gh^{-1}\} & \xrightarrow{f_{gh^{-1}}} & Y \times \{gh^{-1}\}.
\end{array}
\]
Therefore we have that for $g, h \in G$ and $\alpha \in H^*(Y^g; \mathbb{Z})$
\[
(h^{-1})^* f_g^* \alpha, h^{-1}gh
\]
and this implies that the map $f$ is $G$-equivariant.

**Corollary (2.4).** If the inclusion maps in homology $f_{g,*}: H_*(Y^g; \mathbb{Z}) \to H_*(Y; \mathbb{Z})$ are injective for all $g \in G$, then the map
\[
f: H^*_\text{virt}(Y, G; \mathbb{Z}) \to H^*(Y; \mathbb{Z})[G]
\]
is an injective homomorphism of rings. Then the ring $H^*(Y, G; \mathbb{Z})$ can be calculated as the subring $f(H^*_\text{virt}(Y, G; \mathbb{Z}))$ of $H^*(Y; \mathbb{Z})[G].$

**Proof.** The pushforward $f_{g,*}: H^*(Y^g; \mathbb{Z}) \to H^*(Y; \mathbb{Z})$ in cohomology can be defined as the composition of the maps $PD \circ f_{g,*} \circ PD^{-1}$ where $PD: H_*(Y; \mathbb{Z}) \cong H^*(Y; \mathbb{Z})$ and $PD_g: H_*(Y^g; \mathbb{Z}) \cong H^*(Y^g; \mathbb{Z})$ are the Poincaré duality isomorphisms. It follows that the maps $f_{g,*}$ are injective.

The above corollary will allow us to calculate the virtual cohomology of a large family of orbifolds, as in the following example.

**Example (2.5).** Consider the action of $\mathbb{Z}/p$ on the complex projective space $\mathbb{C}P^n$
\[
\mathbb{C}P^n \times \mathbb{Z}/p \to \mathbb{C}P^n
\]
\[([z_0 : \cdots : z_n], \lambda^i) \mapsto [z_0 : \cdots : z_{n-1} : \lambda^iz_n]
\]
where the elements of $\mathbb{Z}/p$ are taken as $p$-th roots of unity. For $i \neq 0$ one has that the fixed point set of $\lambda^i$ is
\[
(\mathbb{C}P^n)^{\lambda^i} \cong \mathbb{C}P^{n-1} \cup \{\ast\}
\]
and therefore if we only consider the connected component of $\mathbb{C}P^{n-1}$ then the maps $f_{\lambda^i,*}$ are all injective. The pushforward of the inclusions are
\[
f_{\lambda^i}: H^*(\mathbb{C}P^{n-1}; \mathbb{Z}) = \mathbb{Z}[y]/(y^n) \to H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})
\]
\[y^i \mapsto x^{i+1},
\]
and
\[ f_{\lambda^1}: H^*(\{\ast\}; \mathbb{Z}) = \mathbb{Z}\langle z \rangle \to H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[x]/\langle x^{n+1} \rangle, \]
\[ z \mapsto x^n. \]

Therefore we have
\[ H^*_\text{virt}(\mathbb{C}P^n, \mathbb{Z}/p; \mathbb{Z}) \cong \left( \mathbb{Z}[x]/\langle x^{n+1} \rangle \langle 1 \rangle \oplus \bigoplus_{i=1}^{p-1} (x\mathbb{Z}[x]/\langle x^{n+1} \rangle \oplus \mathbb{Z}(z) \langle \lambda^i \rangle) \right), \]
where \(zx = 0\) and \(z^2 = 0\).

If we take a closer look at the virtual cohomology generated by the inclusions of the \(\mathbb{C}P^{n-1}\)'s, we can see that its elements are truncated polynomials of maximum degree \(n\), whose coefficients are elements in the group ring \(\mathbb{Z}[\mathbb{Z}/p]\) except for the constant term that should be an integer, i.e.
\[ \{P(x) \in \mathbb{Z}[\mathbb{Z}/p][x]/\langle x^{n+1} \rangle | P(0) \in \mathbb{Z}\} \]
where the ring structure is given by multiplication of polynomials.

If we add the classes coming from the inclusions of the points \(\ast\) we obtain that
\[ H^*_\text{virt}(\mathbb{C}P^n, \mathbb{Z}/p; \mathbb{Z}) \cong \{P(x) \in \mathbb{Z}[\mathbb{Z}/p][x]/\langle x^{n+1} \rangle | P(0) \in \mathbb{Z}\} \oplus \bigoplus_{i=1}^{p-1} \mathbb{Z}(z) \langle \lambda^i \rangle/(xz, z^2). \]

Now, as the group \(\mathbb{Z}/p\) is abelian and its action can be factored through an action of \(S^1\), we have that
\[ H^*_\text{virt}(\mathbb{C}P^n, \mathbb{Z}/p; \mathbb{R}) = H^*_\text{virt}(\mathbb{C}P^n, \mathbb{Z}/p; \mathbb{R}). \]
Then
\[ H^*_\text{virt}(\mathbb{C}P^n/\mathbb{Z}/p; \mathbb{R}) \cong \{P(x) \in \mathbb{R}[\mathbb{Z}/p][x]/\langle x^{n+1} \rangle | P(0) \in \mathbb{R}\} \oplus \bigoplus_{i=1}^{p-1} \mathbb{R}(z) \langle \lambda^i \rangle/(xz, z^2). \]

We have seen that for the case in which the homomorphisms \(f^*_g\) are injective, the virtual cohomology is isomorphic to the subring \(f(H^*_\text{virt}(Y, G; \mathbb{Z}))\) of the group ring \(H^*(Y; \mathbb{Z})(G)\). In what follows we will find a set of generators for \(f(H^*_\text{virt}(Y, G; \mathbb{Z}))\).

Let \(H^*(Y; \mathbb{Z})[1_G]\) be the set of elements of the group ring whose label is the identity \(1_G\) of the group \(G\).

**Proposition (2.6).** Suppose that all the homomorphisms \(f^*_g\) are injective and all the \(f^*_g\) are surjective. Denote by \(1_g \in H^0(Y^g; \mathbb{Z})\) the identity of the ring \(H^*(Y^g; \mathbb{Z})\). Then the set
\[ W := H^*(Y; \mathbb{Z})[1_G] \cup \{f_g^*(1_g)|g| \in G\} \]
generates the ring \(f(H^*_\text{virt}(Y, G; \mathbb{Z})). \)

**Proof:** It is clear that \(W \subset f(H^*_\text{virt}(Y, G; \mathbb{Z}))\); we need to prove that for any \(a \in H^*(Y^g; \mathbb{Z})\) the element \((f^*_g(a))g\) can be generated with elements in \(W\).

We know that the pullback \(f^*_g\) is surjective. Therefore there exists \(b \in H^*(Y; \mathbb{Z})\) such that \(f^*_g b = a\). By the module structure of the pushforward we have
\[ f^*_g(a) = f^*_g(1_g a) = f^*_g(1_g f^*_g b) = (f^*_g 1_g) b \]
which implies that in the group ring
\[ (1_g b 1_G) = ((f^*_g 1_g) b) g = (f^*_g a) g. \]
\[ \square \]
3. Symmetric Product

It was shown in [LUX07] that for an even dimensional compact and closed manifold $M$, the virtual cohomology of the orbifold $[M^n / S_n]$ is a subring of the string homology of the loop orbifold of the symmetric product (see [LUX]).

In this section we will calculate the virtual cohomology of the pair $(M^n, S_n)$ in terms of the cohomology of $M$. As we will make use of the Kunneth isomorphism we will restrict to real coefficients. We would like to remark that if the manifold has torsion free homology, all the calculations that follow can be done with integer coefficients.

So, abusing the notation, we will talk indistinctly of the rings $H^*(M^k; \mathbb{R})$ and $H^*(M; \mathbb{R})^\otimes k$.

We know that the diagonal inclusion $\Delta: M \rightarrow M \times M$ induces an injection $\Delta^*: H^*(M; \mathbb{R}) \rightarrow H^*(M \times M; \mathbb{R})$, and as $\Delta^*(a \otimes 1) = a$ we have that the pullback $\Delta^*$ is surjective. For $\tau \in S_n$ the map $f_\tau: (M^n)^\tau \rightarrow M^n$ is a composition of diagonal maps, so we have that $f_\tau^*$ is injective and that $f_\tau^*$ is surjective. We can therefore apply proposition (2.6) to the pair $(M^n, S_n)$ to get a set of generators.

In what follows we will show that we can reduce the set of generators by only considering the transpositions.

**Lemma (3.1).** Let $\delta: M \rightarrow M^k$ be the diagonal inclusion and let

$$\alpha^k_i: M^{k-1} \rightarrow M^k$$

$$(x_1, \ldots, x_{k-1}) \mapsto (x_1, \ldots, x_{i-1}, x_i, x_i, x_i, x_i, x_{i+1}, \ldots, x_{k-1})$$

be the inclusion that repeats the $i$-th coordinate. Then in cohomology

$$\delta 1 = \prod_{j=1}^{k-1} (\alpha^k_j 1).$$

**Proof.** We will proceed by induction on $k$. When $k = 2$ the formula is true because $\delta = \alpha^2_1$. Assume that we have shown the formula for $k = n$ and let's try to show it for $k = n + 1$. Consider the following diagram of inclusions

$$\begin{align*}
M^n & \xrightarrow{\alpha^{n+1}_1} M^{n+1} \\
\delta & \downarrow \\
M & \xrightarrow{\delta'}
\end{align*}$$

and using the properties of the pushforward we have,

$$\delta' 1 = \alpha^{n+1}_1(\delta 1)$$

$$= \alpha^{n+1}_1 \left( (\alpha^n_1 1)^* (\delta 1) \otimes 1 \right)$$

$$= (\alpha^{n+1}_1 1)(\delta 1) \otimes 1$$

$$= (\alpha^{n+1}_1 1) \left( \prod_{j=1}^{n} (\alpha^{n}_j 1) \otimes 1 \right)$$
By the induction hypothesis the lemma follows.

If we take the cycle $\alpha = (k, k - 1, \ldots, 2, 1) \in \mathfrak{S}_n$ and the transpositions $\tau_i = (i, i + 1)$ we have that $f_{\alpha^1} 1 = \delta_i 1$ and $f_{\tau_i 1} = \alpha^{i}_1 1$. By lemma (3.1) we can conclude that for the cycle $\alpha$ of size $k$ which is the composition of $k - 1$ transpositions $\tau_1 \ldots \tau_{k-1}$ we have that

$$f_{\alpha^1} 1 = \prod_{i=1}^{k-1} (f_{\tau_i 1} 1).$$

Therefore we can reduce the set of generators of the virtual cohomology of the pair $(M^n, \mathfrak{S}_n)$ by considering only the transpositions.

**Proposition (3.2).** The ring $f(H^*_\text{vir}(M^n, \mathfrak{S}_n; \mathbb{R}))$ is generated by the set

$$W = H^*(M^n; \mathbb{R})[1_{\mathfrak{S}_n}] \cup \{f_{\tau_i 1} | \tau \in \mathfrak{S}_n \text{ is a transposition}\}$$

as a subring of $H^*(M^n; \mathbb{R})[\mathfrak{S}_n]$.

If $H^*(M; \mathbb{Z})$ is torsion free, the same result holds but with integer coefficients.

**Example (3.3).** Let’s consider the pair $(M^n, \mathfrak{S}_n)$ with $M = \mathbb{C}P^m$. We therefore have that

$$H^*(M^n; \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]/\langle x_1^{m+1}, \ldots, x_n^{m+1} \rangle$$

and we only need to calculate the pushforward of the diagonal inclusion $\Delta : M \to M \times M$.

By the Kunneth isomorphism we have that

$$H_*(\mathbb{C}P^m \times \mathbb{C}P^m; \mathbb{Z}) \cong H_*(\mathbb{C}P^m; \mathbb{Z}) \otimes H_*(\mathbb{C}P^m; \mathbb{Z})$$

and $H^*(\mathbb{C}P^m \times \mathbb{C}P^m; \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2]/\langle x_1^{m+1}, x_2^{m+1} \rangle$. Let’s take $H^*(\mathbb{C}P^m; \mathbb{Z}) \cong \mathbb{Z}[y]/\langle y^{m+1} \rangle$ and denote by $[\mathbb{C}P^i] \in H_{2i}(\mathbb{C}P^m; \mathbb{Z})$ the generator of the homology in degree $2i$ given by the inclusion $\mathbb{C}P^i \to \mathbb{C}P^m, [z_0 : \cdots : z_i] \mapsto [z_0 : \cdots : z_i : 0 : \cdots : 0]$.

By Poincaré duality (see [Hat02, page 213]) we know that the homology class $[\mathbb{C}P^i]$ is dual to the cohomology class $y^{m-i}$. Now, $\Delta^* x_1 = \Delta^* x_2 = y$ implies that $\Delta^* x_1 x_2^{m-j} = y^m$, and as the classes $\{x_1 x_2^{m-j} | 0 \leq j \leq m\}$ generate the degree $2m$ cohomology of $\mathbb{C}P^m \times \mathbb{C}P^m$, we have by Poincaré duality that

$$\Delta_! [\mathbb{C}P^m] = \sum_{j=0}^{m} [\mathbb{C}P^j] \otimes [\mathbb{C}P^{m-j}].$$

Therefore we can conclude that $\Delta_! 1 = \sum_{j=0}^{m} x_1^j x_2^{m-j}$.

Then, for the transposition $\tau = (k, l)$ we get that

$$f_{\tau 1} = \sum_{j=0}^{m} x_k^j x_l^{m-j}.$$
and we can conclude that \( H^*_\text{virt}((\mathbb{C}P^m)^n, \mathcal{G}_n, \mathbb{Z}) \) is isomorphic to the subring of \( \mathbb{Z}[x_1, \ldots, x_n]/(x_1^{m+1}, \ldots, x_n^{m+1})[\mathcal{G}_n] \) generated by

\[
W = \{ (1 \delta_{i,x}) \cup (x_i \delta_{i,x}) \mid 1 \leq i \leq n \} \cup \left\{ \sum_{j=0}^{m} x_k x_l^{m-j} (k, l) \mid 1 \leq k < l \leq n \right\}
\]

**Example (3.4).** Let’s pay particular attention to the case on which \( n = 2 \) and \( M \) is a connected, differentiable, compact and closed manifold of dimension \( d \). Denote by \( \Omega \in H^d(M; \mathbb{R}) \) the generator of the top cohomology, then we have that

\[
(\Delta, 1)(\Delta, 1) = \chi(M) \Omega \otimes \Omega
\]

where \( \chi(M) \) is the Euler number of \( M \) (see [BT82], Pro. 11.24). Moreover, by the properties of the pushforward in cohomology we have that for any \( \alpha, \beta \in H^*(M; \mathbb{R}) \)

\[
(\Delta, 1)(\alpha \otimes \beta) = \Delta((\Delta^*(\alpha \otimes \beta))) = \Delta(\alpha \beta),
\]

hence, if \( \alpha_1 \beta_1 = \alpha_2 \beta_2 \) we have that

\[
(\Delta, 1)(\alpha_1 \otimes \beta_1) = (\Delta, 1)(\alpha_2 \otimes \beta_2)
\]

and if \( \deg(\alpha) + \deg(\beta) > d \)

\[
(\Delta, 1)(\alpha \otimes \beta) = 0.
\]

If we consider the ring

\[
H^*(M; \mathbb{R})^\otimes \langle u \rangle
\]

where \( u \) represents the element \( (\Delta, 1) \) then we can see that \( u^3 = 0 \) and \( u^2 - \chi(M) \Omega \otimes \Omega \). The annihilator ideal of \( u \) is generated by the elements \( \alpha \otimes \beta \) where \( \deg(\alpha) + \deg(\beta) > d \), and by the elements \( (\alpha_1 \otimes \beta_1 - \alpha_2 \otimes \beta_2) \) where \( \alpha_1 \beta_1 = \alpha_2 \beta_2 \).

In the case that \( M = \mathbb{C}P^m \) we can see that \( H^*_\text{virt}((\mathbb{C}P^m)^2, \mathcal{G}_2; \mathbb{Z}) \) is isomorphic to

\[
\mathbb{Z}[x, y, u]/\langle x^{m+1}, y^{m+1}, u^2 - (m + 1)x y^m, u(x - y) \rangle
\]

where \( u^3 = 0 \) because \( u^3 = (m + 1)x y^m u = (m + 1)x^{m-1} y^{m+1} u = 0 \).

In the case that \( m = 1 \) we have that \( H^*_\text{virt}((\mathbb{C}P^1)^2, \mathcal{G}_2; \mathbb{Z}) \) is

\[
\mathbb{Z}[x, y, u]/\langle x^2, y^2, u^2 - 2xy, u(x - y) \rangle.
\]

The \( \mathbb{Z}/2 \) invariants are generated as an \( \mathbb{R} \)-module by \( x + y, u, xy \) and \( ux \). Therefore, if we take \( w = x + y \), then \( w^2 = 2xy = u^2 \), \( 2ux = uw \) and \( w^3 = 0 \). So we can conclude that

\[
H^*_\text{virt}(((\mathbb{C}P^1)^2/\mathcal{G}_2); \mathbb{R}) \cong \mathbb{R}[w, u]/\langle w^3, u^2, u^2 - w^2 \rangle.
\]

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