A Galoisian approach to SUSY QM and beyond

Primitivo B. Acosta-Humáñez

Department of Mathematics and Statistics
Universidad del Norte
pacostahumanez@uninorte.edu.co

Quantum Integrable Systems, Universidad de los Andes
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Galois Theory for Linear Differential Equations

**Differential field:** \((K, d/dt)\), field with a derivation.

**Example:** \(K = \) field of meromorphic functions, \(M(\Gamma)\), over a Riemann surface \(\Gamma\) (classical particular case: rational functions \(\mathbb{C}(t) = \) meromorphic functions of the Riemann sphere \(\mathbb{P}^1\)).

**Linear differential equation with coefficients in** \(K\):

\[
\frac{d\xi}{dt} = A\xi, \quad A \in \text{Mat}(m, K).
\]  

**Picard-Vessiot extension of** \((1)\): \(L := K(u_{ij})\), \(U := (u_{ij})\): fundamental matrix associated to \((1)\) (\(K \subset L \sim \) Galois extension of a polynomial).
Differential automorphism $\sigma : L \rightarrow L$: 
\begin{enumerate}
  \item automorphism of the field $L$, 
  \item $\sigma d/dt = d/dt\sigma$. 
\end{enumerate}

Galois group of the equation (1):
\[ G := Gal(1) = Gal(L/K) = \{\sigma : L \rightarrow L : \sigma \text{ differential automorphism } \sigma_K = Id\} \]
($\sim$ Galois group of a polynomial).
$G$ is the transformation group preserving all the algebraic relations of $u_{ij}$ with coefficients in $K$.

**Theorem**

$G$ is a linear algebraic subgroup of $GL(m, \mathbb{C})$.

(1) integrable: the extension $L$ is obtained from $K$ by a combination of algebraic extensions, quadratures and exponentials of quadratures.

**Theorem**

(1) is integrable $\iff G^0$ is solvable.
Kovacic Algorithm

This is an effective algorithm to compute the solutions of differential equations:

\[ \partial_x^2 y = r y, \quad r \in \mathbb{C}(x). \]

There are four cases in Kovacic’s algorithm. Only for cases 1, 2 and 3 we can solve the differential equation, but for the case 4 the differential equation is not integrable.

Theorem

Let \( G \) be an algebraic subgroup of \( SL(2, \mathbb{C}) \). Then, up to conjugation, one of the following cases occurs.

1. \( G \subseteq B := \text{triangular unimodular group} \).
2. \( G \not\subseteq B, \ G \subseteq D_\infty := \text{infinite dihedral group} \).
3. \( G \in \{ A_4^{SL_2}, S_4^{SL_2}, A_5^{SL_2} \} \).
4. \( G = SL(2, \mathbb{C}) \).
Algebrization Procedure

Differential equations with non-rational coefficients into
differential equations with rational coefficients.
A change of variable $z = z(x)$ is called hamiltonian when
$(z(x), \partial_x z(x))$ is a solution curve of the hamiltonian system
\[
\begin{align*}
\partial_x z &= \partial_w H \\
\partial_x w &= -\partial_z H
\end{align*}
\]
with $H = H(z, w) = \frac{w^2}{2} + V(z)$.

Theorem [Algebrization of Schrödinger Operator]

$H = -\partial_x^2 + V$ is algebrizable through a hamiltonian change
of variable $z = z(x)$ if and only if there exists $\hat{V}, \alpha$ such that
\[
\frac{\partial_z \alpha}{\alpha}, \frac{\hat{V}}{\alpha} \in \mathbb{C}(z), \text{ where } V(x) = \hat{V}(z(x)), \alpha(z) = (\partial_x z)^2.
\]

The algebrization of $H$ is given by the operator
\[
\hat{H} = -\left(\alpha \partial_z + \frac{1}{2} \partial_z \alpha\right) \partial_z + \hat{V}(z).
\]
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Non-relativistic quantum mechanics

Schrödinger equation

Consider the one-dimensional stationary non-relativistic Schrödinger equation

\[ H\Psi = \lambda\Psi, \quad H = -\frac{\partial^2}{\partial x^2} + V. \]

SUSY QM. A supersymmetric quantum mechanical system is one in which there are operators \( Q_i \) and \( \mathcal{H} \) satisfying

\[ [Q_i, \mathcal{H}] = 0, \quad \{Q_i, Q_j\} = \delta_{ij}\mathcal{H}, \quad \{Q_i, Q_j\} = Q_iQ_j + Q_jQ_i \]

For \( n = 2 \), the supercharges \( Q_i \) are defined as

\[ Q_{\pm} = \frac{\sigma_1 p \pm \sigma_2 W(x)}{2}, \quad Q_+ = Q_1, \quad Q_- = Q_2, \quad p = -i\hbar\partial_x, \]

where \( W \) is the superpotential and \( \sigma_i \) the Pauli spin matrices.

The operator \( \mathcal{H} \), satisfying \( Q_i\mathcal{H} = \mathcal{H}Q_i \) and \( 2Q_i^2 = \mathcal{H} \), is

\[ \mathcal{H} = \frac{l_2 p^2 + l_2 W^2(x) + \hbar\sigma_3 \partial_x W(x)}{2} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad l_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]
Motivation: Known shape invariant potentials in Quantum Mechanics

<table>
<thead>
<tr>
<th>Potential</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2} m \omega^2 \left(x - \sqrt{\frac{2}{m}} \frac{b}{\omega}\right)^2$</td>
<td>Shifted H. O.</td>
</tr>
<tr>
<td>$\frac{1}{2} m \omega^2 r^2 + \frac{l(l+1)\hbar^2}{2mr^2} - \left(l + \frac{3}{2}\right) \hbar \omega$</td>
<td>3D H.O.</td>
</tr>
<tr>
<td>$- \frac{e^2}{r} + \frac{l(l+1)\hbar^2}{2mr^2} - \frac{me^4}{2(l+1)^2\hbar^2}$</td>
<td>Coulomb</td>
</tr>
<tr>
<td>$A^2 + B^2 e^{-2ax} - 2B \left(A + \frac{a\hbar}{2\sqrt{2m}}\right) e^{-ax}$</td>
<td>Morse 1</td>
</tr>
<tr>
<td>$A^2 + \frac{B^2 - A^2 - \frac{Aa\hbar}{\sqrt{2m}}}{\cosh^2 ax} + \frac{B \left(2A + \frac{a\hbar}{\sqrt{2m}}\right) \sinh ax}{\cosh^2 ax}$</td>
<td>Morse 2</td>
</tr>
<tr>
<td>$A^2 + \frac{B^2}{A^2} + 2B \tanh ax - A \frac{A + \frac{a\hbar}{\sqrt{2m}}}{\cosh^2 ax}$</td>
<td>Rosen-Morse 1</td>
</tr>
<tr>
<td>$A^2 + \frac{B^2 + A^2 + \frac{Aa\hbar}{\sqrt{2m}}}{\sinh^2 ar} - \frac{B \left(2A + \frac{a\hbar}{\sqrt{2m}}\right) \cosh ar}{\sinh^2 ar}$</td>
<td>Rosen-Morse 2</td>
</tr>
<tr>
<td>$A^2 + \frac{B^2}{A^2} - 2B \coth ar + A \frac{A - \frac{a\hbar}{\sqrt{2m}}}{\sinh^2 ar}$</td>
<td>Eckart 1</td>
</tr>
<tr>
<td>$-A^2 + \frac{B^2 + A^2 - \frac{Aa\hbar}{\sqrt{2m}}}{\sin^2 ax} - \frac{B \left(2A - \frac{a\hbar}{\sqrt{2m}}\right) \cos ax}{\sin^2 ax}$</td>
<td>Eckart 2</td>
</tr>
<tr>
<td>$-(A + B)^2 + \frac{A \left(A - \frac{a\hbar}{\sqrt{2m}}\right)}{\cos^2 ax} + \frac{B \left(B - \frac{a\hbar}{\sqrt{2m}}\right)}{\sin^2 ax}$</td>
<td>Pöschl-Teller 1</td>
</tr>
<tr>
<td>$(A - B)^2 - \frac{A \left(A + \frac{a\hbar}{\sqrt{2m}}\right)}{\cosh^2 ar} + \frac{B \left(B - \frac{a\hbar}{\sqrt{2m}}\right)}{\sinh^2 ar}$</td>
<td>Pöschl-Teller 2</td>
</tr>
</tbody>
</table>

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Notations

\[ \mathcal{L}_\lambda := H\Psi = \lambda \Psi, \quad H = -\partial_x^2 + V(x), \quad V \in K. \]

\( \Lambda \subseteq \mathbb{C} \): set of eigenvalues \( \lambda \) such that \( \mathcal{L}_\lambda \) is integrable.

\( \Lambda_+ := \{ \lambda \in \Lambda \cap \mathbb{R} : \lambda \geq 0 \} \), \( \Lambda_- := \{ \lambda \in \Lambda \cap \mathbb{R} : \lambda \leq 0 \} \).

\( L_\lambda \): Picard-Vessiot extension of \( \mathcal{L}_\lambda \).

\( \text{Gal}(L_\lambda/K) \): differential Galois group of \( \mathcal{L}_\lambda \).

The set \( \Lambda \) will be called the algebraic spectrum (or alternatively the Liouvillian spectral set) of \( H \).

\( \Lambda \) can be \( \emptyset \), i.e., \( \text{Gal}(L_\lambda/K) = SL(2, \mathbb{C}) \ \forall \lambda \in \mathbb{C} \).

If \( \lambda_0 \in \Lambda \) then \( (\text{Gal}(L_{\lambda_0}/K))^0 \subseteq \mathbb{B} \).
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Theorem (Polynomial potentials and Galois groups)

Let \( V(x) \in \mathbb{C}[x] \) a polynomial of degree \( k > 0 \). Then

1. \( \text{Gal}(L_\lambda/K) = \text{SL}(2, \mathbb{C}) \), or,
2. \( \text{Gal}(L_\lambda/K) = \mathbb{B} \).

Corollary

Assume that \( V(x) \) is an algebraically solvable polynomial potential. Then \( V(x) \) is of degree 2.

Remark. When a polynomial potential is algebraically solvable or quasi-solvable, then the Galois group of the Schrödinger equation is exactly the Borel group (triangular).
Rational Potentials and Kovacic’s Algorithm

Three dimensional harmonic oscillator potential

\[ V(r) = r^2 + \frac{\ell(\ell + 1)}{r^2} - (2\ell + 3), \quad \ell \in \mathbb{Z}. \]

The Schrödinger equation is

\[ \frac{\partial^2}{\partial r^2} \psi = \left( r^2 + \frac{\ell(\ell + 1)}{r^2} - (2\ell + 3) - \lambda \right) \psi. \]

**Algebraic Spectrum** Applying Kovacic’s algorithm we obtain \( \Lambda = 2\mathbb{Z} \).

**Bound states.**

\[ \psi_n(r) = r^{\ell+1} P_{2n}(r) e^{-\frac{r^2}{2}}, \quad \lambda \in 4\mathbb{N}. \]

**Galois groups.** For \( \lambda \in 4\mathbb{Z} \) we have that \( \text{Gal}(L_\lambda/K) = \mathbb{B} \).
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Algebrized Schrödinger equations. In some cases we can algebrize $H\Psi = \lambda \Psi$ to apply Kovacic’s algorithm, etc...

**Theorem (Algebrization Algorithm for the exponential case)**

Consider $H$ with $V(x) = g(z_1(x), \cdots, z_n(x))$, $z_i(x) = e^{\mu_i x}$, $\mu_i \in \mathbb{C}^*$. The operator $H$ is algebrizable if and only if

$$\frac{\mu_i}{\mu_j} \in \mathbb{Q}^*, \quad 1 \leq i \leq n, \ 1 \leq j \leq n, \quad g \in \mathbb{C}(z).$$

There exists $\mu \in \mathbb{C}^*$ such that $\mu_i = c_i \mu$, where $c_i \in \mathbb{Q}^*$. One Hamiltonian change of variable is $z = e^{\mu x} q$, where

$$c_i = \frac{p_i}{q_i}, \quad p_i, q_i \in \mathbb{Z}^*, \quad \gcd(p_i, q_i) = 1 \quad \text{and} \quad q = \text{lcm}(q_1, \cdots, q_n).$$

The algebrized Schrödinger equation of $H\Psi = \lambda \Psi$ is

$$\frac{\partial^2 \hat{\Psi}}{\partial z^2} + \frac{1}{z} \frac{\partial \hat{\Psi}}{\partial z} - q^2 \frac{g(z^{m_1}, \cdots, z^{m_n}) - \lambda}{\mu^2 z^2} \hat{\Psi} = 0, \quad m_i = \frac{q p_i}{q_i}.$$
The Morse Potential. $V(x) = e^{-2x} - e^{-x}$.

The Schrödinger equation $H\psi = \lambda\psi$ is

$$\partial_x^2 \psi = (e^{-2x} - e^{-x} - \lambda) \psi.$$

By the Hamiltonian change of variable $z = z(x) = e^{-x}$, we obtain

$$\alpha(z) = z^2, \quad \hat{V}(z) = z^2 - z.$$

Thus, $\hat{K} = \mathbb{C}(z)$ and $K = \mathbb{C}(e^x)$. In this way, the algebrized Schrödinger equation $\hat{H}\hat{\psi} = \lambda\hat{\psi}$ is

$$z^2 \partial_z^2 \hat{\psi} + z\partial_z \hat{\psi} - (z^2 - z - \lambda)\hat{\psi} = 0.$$

Algebraic Spectrum and Galois Group.

$\Lambda = \{-n^2 : n \geq 0\} = \text{spec}_p(H)$. $Gal(L_0/K) = \mathbb{B}$, $Gal(L_n/K) = \mathbb{G}_m$. 
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Recent and further research involving P-V Theory and QM

Y. Brezhnev on finite gap potentials by means of Picard-Vessiot theory
D. Blazquez-Sanz and K. Yagasaki on the Sturm-Liouville problems
J.P. Ramis et al, concerning to Stokes phenomena
T. Stachowiak on the Dirac equations
T. Dreyfuss on the parametric Galois groups in Quantum Mechanics

In progress there are some projects in Quantum Field Theory, Atomic and Molecular Physics, etc...
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