A brief introduction to Montgomery Conjecture (Pair correlation of zeros of $\zeta$)

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1. Outline

2. Introducing \( \zeta \)

3. Montgomery conjecture

4. GUE

5. Some ideas around Montgomery conjecture
Outline

Introducing $\zeta$
Montgomery conjecture
GUE
Some ideas around Montgomery conjecture
\( \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \) is defined for \( s = \sigma + i\gamma \) for \( \sigma > 1 \).
To extend $\zeta$ meromorphically to $\mathbb{C}$ we use the formula:

$$
\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \left\{ \frac{1}{s(s - 1)} + \int_{1}^{\infty} \left( x^{1/2s-1} + x^{-1/2s-1} \right) \sum_{n=1}^{\infty} e^{-n^2 \pi x} \, dx \right\}
$$

and observe the right–hand side integral represents an entire function of $s$. 

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To prove the previous formula we use

$$\Gamma\left(\frac{S}{2}\right) = \int_0^\infty e^{-t} t^{s/2-1} dt = n^s \pi^{s/2} \int_0^\infty e^{-n^2 \pi x} x^{1/2s-1} dx.$$
Some properties of $\zeta$
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$\zeta(s) \neq 0$ for $\sigma > 1$. 
Some properties of $\zeta$

$\zeta(s) \neq 0$ for $\sigma > 1$. This follows from the convergence of the Euler product formula:

$$
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left( 1 - \frac{1}{p^s} \right).
$$
Some properties of $\zeta$

$\zeta(s)$ has simple zeros at $0, -2, -4, \ldots$. 
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$\zeta(s)$ has simple zeros at $0, -2, -4, \cdots$. Because $\Gamma(s/2)$ has simple poles at $0, -2, -4, \cdots$ and

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \left\{ \frac{1}{s(s - 1)} + \int_1^\infty (x^{1/2s - 1} + x^{-1/2s - 1}) \sum_{n=1}^\infty e^{-n^2 \pi x} \, dx \right\}.$$
Some properties of $\zeta$

$\zeta(s) \neq 0$ for $\sigma = 1$ (result of Hadamard and De la Vallée Poussin).
Some properties of $\zeta$

Zeros of $\zeta$ are symmetric respect to $\sigma = 1/2$ for $0 \leq \sigma \leq 1$. 
Some properties of $\zeta$

Zeros of $\zeta$ are symmetric respect to $\sigma = 1/2$ for $0 \leq \sigma \leq 1$. Because

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{-\frac{(1-s)}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

Hence $\zeta(s) \neq 0$ for $\sigma = 0$. 
Some properties of $\zeta$

From
$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \left\{ \frac{1}{s(s-1)} + \int_1^{\infty} \left( x^{1/2s-1} + x^{-1/2s-1} \right) \sum_{n=1}^{\infty} e^{-n^2 \pi x} \, dx \right\}$$
we can see zeros of $\zeta$ are symmetric respect the real axis because conjugates of zeros are also zeros.
Some properties of $\zeta$

The zeros of $\zeta$ are symmetric respect to $s = 1/2$. 
The zeros of $\zeta$ are symmetric respect to $s = 1/2$. Because 
$$\xi(s) = \frac{1}{2} s(s - 1) \pi^{-1/2s} \Gamma(s/2) \zeta(s)$$ 
the function $\frac{1}{2} s \Gamma(s/2)$ has no zeros.
Some properties of $\zeta$

**Riemann conjecture:** All the non–trivial zeros of $\zeta$ are contained in the line $\sigma = 1/2$. 
Some properties of $\zeta$

$N(T)$ the number of zeros in the critical line such that, $0 \leq \gamma < T$ then

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log(T)).$$
(1973) Montgomery Pair Correlation Conjecture: Assume the Riemann hypothesis. For fixed $0 < a < b < \infty$ as $T \to \infty$,

$$\sum_{(\gamma, \gamma') \in [0, T]^2: a \leq (\gamma - \gamma') \frac{\log(T)}{2\pi} \leq b} 1 \sim \frac{T}{2\pi} \log(T) \int_a^b \left(1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2\right) du.$$
Gaussian Unitary Ensemble
**Gaussian Unitary Ensemble**

**Definition**

*A Gaussian Unitary Ensemble* is a set of $N \times N$ Hermitian matrices $H := (a_{ij})$ such that:

- The real and imaginary parts of the entries $a_{ij}$ of $H$ are independent random variables.
- $P(H)dH = P(H')dH'$ where $H' = U^{-1}HU$ where $U$ is unitary.
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$H$ is GUE then $a_{ij}$ have Gaussian distributions
GUE pair correlation of eigenvalues
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Let $R(x_1, x_2)dx_1 dx_2$ denotes the pair correlation of eigenvalues. Intuitively the probability that there are pairs of eigenvalues in $[x_1, x_1 + dx_1] \times [x_2, x_2 + dx_2]$
Asymptotics of GUE pair correlation distribution of eigenvalues
Asymptotics of GUE pair correlation distribution of eigenvalues

**Theorem**

Let $R(x_1, x_2)$ denotes the pair correlation of eigenvalues. Then,

$$\frac{1}{\alpha_1 \alpha_2} R(x_1, x_2) \sim 1 - \left( \frac{\sin(\pi u)}{\pi u} \right)^2$$

as $N \to \infty$ where $u = |x_1/\alpha_1 - x_2/\alpha_2|$ and $\alpha_j = \frac{\pi}{\sqrt{2N-x_j^2}}$ is the mean local spacing of eigenvalues at $x_j$, $j = 1, 2$. 
Convolution formula
Some ideas around Montgomery conjecture

Convolution formula

Theorem

We have

$$\sum_{(\gamma, \gamma') \in [0, T]^2} r((\gamma - \gamma') \frac{\log(T)}{2\pi}) \omega(\gamma - \gamma') \sim \frac{T}{2\pi} \log(T) \int_{\infty}^{\infty} F(u) \hat{r}(u) du$$
Observe that if $r(u) = \chi_{[a,b]}(u)$ and $\omega(\gamma - \gamma') \to 1$ when $(\gamma - \gamma') \to 0$ we will have a tool for motivating Montgomery conjecture!
Motivation of M. Conjecture
Motivation of M. Conjecture

Suppose we have already $F$ and have proved the previous convolution formula...
Montgomery conjectured furthermore:

\[ F(\alpha) = \begin{cases} 
1 + o(1) & \text{for } |\alpha| \geq 1 \\
(1 + o(1)) T^{-2|\alpha|} \log(T) + |\alpha| + o(1) & \text{for } |\alpha| < 1.
\]
Motivation of M. Conjecture

$$\sum_{(\gamma,\gamma')\in[0,T]^2} r(\gamma - \gamma') \frac{\log(T)}{2\pi} \omega(\gamma - \gamma')$$
Motivation of M. Conjecture

\[ \sum_{(\gamma, \gamma') \in [0, T]^2} r\left( \frac{(\gamma - \gamma') \log T}{2\pi} \right) \omega(\gamma - \gamma') \sim \frac{T}{2\pi} \log(T) \int_{-\infty}^{\infty} F(u) \hat{r}(u) du \]
Motivation of M. Conjecture

\[
\sum_{(\gamma, \gamma') \in [0, T]^2} r\left(\frac{(\gamma - \gamma') \log(T)}{2\pi}\right) \omega(\gamma - \gamma') \\
\sim \frac{T}{2\pi} \log(T) \int_{\infty}^{\infty} F(u) \hat{r}(u) du \\
= \frac{T}{2\pi} \log(T) \int_{\infty}^{\infty} \hat{F}(u) r(u) du
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Motivation of M. Conjecture

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\sum_{(\gamma,\gamma') \in [0,T]^2} r((\gamma - \gamma') \frac{\log(T)}{2\pi}) \omega(\gamma - \gamma') \\
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= \frac{T}{2\pi} \log(T) \int_\infty^\infty \hat{F}(u) r(u) du
\]

If we had a function \( F \) such that

\[
\hat{F}(u) = 1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2 + \delta_0
\]
Motivation of M. Conjecture

\[ \sum_{(\gamma, \gamma') \in [0, T]^2} r((\gamma - \gamma') \log(T)) \frac{\log(T)}{2\pi} \omega(\gamma - \gamma') \]

\[ \sim \frac{T}{2\pi} \log(T) \int_{\infty}^{\infty} F(u) \hat{r}(u) du \]

\[ = \frac{T}{2\pi} \log(T) \int_{\infty}^{\infty} \hat{F}(u) r(u) du \]

\[ = \frac{T}{2\pi} \log(T) \int_{\infty}^{\infty} (1 - \left( \frac{\sin(\pi u)}{\pi u} \right)^2 + \delta_0) r(u) du. \]
Motivation of M. Conjecture

Finally

\[
\frac{T}{2\pi} \log(T) \int_{\infty}^{\infty} \left( 1 - \left( \frac{\sin(\pi u)}{\pi u} \right)^2 + \delta_0 \right) r(u) du =
\]

\[
\frac{T}{2\pi} \log(T) \int_{a}^{b} \left( 1 - \left( \frac{\sin(\pi u)}{\pi u} \right)^2 \right) du.
\]
Remark
Remark

Using the previous approach for $r(u) := r_1(u) := \frac{\sin(2\pi au)}{\pi au}$, it is possible to prove that 2/3 of the zeros of the critical line are simple.
What is missing
What is missing

- $F$ satisfies convolution formula and
  $$\hat{F}(u) = 1 - \left( \frac{\sin(\pi u)}{\pi u} \right)^2 + \delta_0.$$

- $\omega$.

- Motivates

$$F(\alpha) = \begin{cases} 
1 + o(1) & \text{for } |\alpha| \geq 1 \\
(1 + o(1)) T^{-2|\alpha| \log(T)} + |\alpha| + o(1) & \text{for } |\alpha| < 1. 
\end{cases}$$
What is missing

It is enough to consider $\omega(u) = \frac{4}{4+u^2}$. Roughly because, for $T \gg 0$, $a \leq (\gamma - \gamma') \frac{\log(T)}{2\pi} \leq b$ only if $\gamma - \gamma'$ is small, hence $\omega(\gamma - \gamma') \sim 1$. 
What is missing

\[ F(u) := F(u, T) := \left( \frac{T}{2\pi} \log(T) \right)^{-1} \sum_{(\gamma, \gamma') \in [0, T]^2} T^{iu(\gamma - \gamma')} \omega(\gamma - \gamma'). \]
Proof of convolution formula:
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Proof of convolution formula:

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= \sum_{(\gamma, \gamma') \in [0, T]^2} \omega(\gamma - \gamma') \int_{-\infty}^{\infty} e^{iu(\gamma - \gamma')} \log(T) \hat{r}(u) du
= \sum_{(\gamma, \gamma') \in [0, T]^2} \omega(\gamma - \gamma') r(\alpha(\gamma - \gamma') \frac{\log(T)}{2\pi}). \Box
\]
What is missing
What is missing

Proposition

We have:

\[ \hat{F}(u) = 1 - \left( \frac{\sin(\pi u)}{\pi u} \right)^2 + \delta_0, \]

for \( u < 1 \).
What is missing

Assuming Riemann hypothesis Montgomery proves:

\[ F(u) = (1 + o(1)) T^{-2u} \log(T) + u + o(1), \]

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Since \( T^{-2u} \log(T) \) behaves like \( \delta_0 \) when \( T \to \infty \), we can deduce that in the limit \( F(u) = |u| + \delta_0 \).
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Since \( T^{-2u} \log(T) \) behaves like \( \delta_0 \) when \( T \to \infty \), we can deduce that in the limit \( F(u) = |u| + \delta_0 \).

We know that if \( f(u) := \left( \frac{\sin(\pi u)}{\pi u} \right)^2 \) then \( \hat{f}(u) = (1 - |u|) \chi_1(u) \).

The proposition follows from \( F(u) = (1 - \hat{f}(u)) + \delta_0(u) \) because \( \hat{\delta}_0 = 1 \).
Numerical motivation
Numerical motivation

Odlyzko in 1987 obtained many zeros in the critical line with very high heights to empirically test the Montgomery conjecture.
Thank you!