Top eigenvalue of a random matrix: Large deviations

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THE GENERALISED PRODUCT MOMENT DISTRIBUTION IN SAMPLES FROM A NORMAL MULTIVARIATE POPULATION.

By JOHN WISHART, M.A., B.Sc. Statistical Department, Rothamsted Experimental Station.

1. Introduction.

For some years prior to 1915, various writers struggled with the problems that arise when samples are taken from uni-variate and bi-variate populations, assumed in most cases for simplicity to be normal. Thus “Student,” in 1908, by considering the first four moments, was led by K. Pearson’s methods to infer the distribution of standard deviations, in samples from a normal population. His results, for comparison with others to be deduced later, will be stated in the form

$$dp = \frac{1}{\Gamma\left(\frac{N-1}{2}\right)} \frac{1}{A^{N-1/2}} e^{-\frac{A}{2}} \frac{A^{N/2-2}}{2} da \quad \ldots \ldots (1),$$

where $N$ is the size of the sample, and

$$A = \frac{N}{2\sigma^2}, \quad a = s^2,$$

$\sigma$ being the standard deviation of the sampled population, and $s$ that estimated from the sample. Thus, if $x_1, x_2, \ldots, x_N$ are the sample values,

$$N\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i,$$

and

$$N\bar{x}^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2.$$

When bi-variate populations were considered, other problems arose, such as the distribution of the correlation coefficient and of the regression coefficient in samples. These problems, taken by themselves, were found to be difficult, and only approximative results had been reached, when, in 1915, R. A. Fisher gave a formula for the simultaneous distribution of the three quadratic statistical derivatives, namely the two variances (squared standard deviations) and the product moment coefficient. Thus, let $x_1, x_2, \ldots, x_N$ represent the sample values of the
Covariance Matrix

\[ X = \begin{bmatrix}
1 & X_{11} & X_{12} \\
2 & X_{21} & X_{22} \\
3 & X_{31} & X_{32}
\end{bmatrix} \]

in general

(MxN)

\[ X^t = \begin{bmatrix}
X_{11} & X_{21} & X_{31} \\
X_{12} & X_{22} & X_{32}
\end{bmatrix} \]

in general

(NxM)

\[ W = X^t X = \begin{bmatrix}
X_{11}^2 + X_{21}^2 + X_{31}^2 & X_{11}X_{12} + X_{21}X_{22} + X_{31}X_{32} \\
X_{12}X_{11} + X_{22}X_{21} + X_{32}X_{31} & X_{12}^2 + X_{22}^2 + X_{32}^2
\end{bmatrix} \]

(NxN) COVARIANCE MATRIX (unnormalized)

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Covariance Matrix

\[
X = \begin{bmatrix}
1 & X_{11} & X_{12} \\
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\end{bmatrix}
\]

Null model → random data: \( X \) → random \((M \times N)\) matrix

→ \( W = X^t X \) → random \( N \times N \) matrix (Wishart, 1928)
The problem of the spacing of levels is neither a terribly important one nor have I solved it. That is really the point which I want to make very definitely. As we go up in the energy scale it is evident that the detailed analyses which we have seen for low energy levels is not possible, and we can only make use of the approximate theory that the spaced levels are distributed according to the Poisson law or some other law.

Let me say only one more word. It is very likely that the curve in Figure I is a universal function. In other words, it does not depend on the details of the model with which you are working. There is one particular model in which the probability of the energy levels can be written down exactly. I mentioned this distribution already at Gatlinburg. It is called the Wishart distribution. Consider a set of symmetric matrices in such a way that the diagonal element $m_{11}$ has a distribution $\exp(-m_{11}^2/4)$. In other words, the probability that this diagonal element will assume the value $m_{11}$ is proportional to $\exp(-m_{11}^2/4)$. Then as I mentioned, and this was shown a long time ago by Wishart, the probability for the characteristic roots to be $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$, if $\Sigma$ is an $n$ dimensional matrix, is given by the expression:

$$\frac{1}{2^n} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \left( (\lambda_1 - \lambda_2) (\lambda_2 - \lambda_3) \cdots (\lambda_{n-1} - \lambda_n) \right)^\alpha \lambda_n^{\alpha_n}.$$

The probability that two successive roots have a distance $\chi$, then you have to integrate over all of them except two. This is very easy to do for the first integration, possible to do for the second integration, but when you get to the third, fourth and fifth, etc., integrations you have the same problem as in statistical mechanics, and presumably the solution of the problem will be accomplished by one of the methods of statistical mechanics. Let me only mention that I did integrate over all of them except one, and the result is $1 - \frac{1}{2\pi} \sqrt{4n - 3\lambda^2}$. This is the probability that the root shall be $\lambda$. All I have to do is to integrate over one less variable than I have integrated over, but this I have not been able to do so far.

**DISCUSSION**

**W. Havens:** Where does one find out about a Wishart distribution?
the root shall be \( \lambda \). All I have to do is to integrate over one less variable than I have integrated over, but this I have not been able to do so far.

**DISCUSSION**

W. HAVENS: Where does one find out about a Wishart distribution?

E. WIGNER: A Wishart distribution is given in S. S. Wilks book about statistics and I found it just by accident.
spectra of heavy nuclei

238
U

232
Th

WIGNER (’50) : replace complex H by random matrix
DYSON, GAUDIN, MEHTA, .....
Applications of Random Matrices

**Physics:** nuclear physics, quantum chaos, disorder and localization, mesoscopic transport, optics/lasers, quantum entanglement, neural networks, gauge theory, QCD, matrix models, cosmology, string theory, statistical physics (growth models, interface, directed polymers...), ....

**Mathematics:** Riemann zeta function (number theory), free probability theory, combinatorics and knot theory, determinantal points processes, integrable systems, ...

**Statistics:** multivariate statistics, principal component analysis (PCA), image processing, data compression, Bayesian model selection, ...

**Information Theory:** signal processing, wireless communications, ..

**Biology:** sequence matching, RNA folding, gene expression network ...

**Economics and Finance:** time series analysis,....

**Recent Ref:** The Oxford Handbook of Random Matrix Theory
Working model: real, symmetric $N \times N$ Gaussian random matrix

$$J = \begin{pmatrix} J_{11} & J_{12} & \ldots & J_{1N} \\ J_{12} & J_{22} & \ldots & J_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ J_{1N} & J_{2N} & \ldots & J_{NN} \end{pmatrix}$$

$$\text{Prob.}[J] \propto \exp \left[ -\frac{1}{2} \sum_{i,j} J_{ij}^2 \right] = \exp \left[ -\frac{1}{2} \text{Tr}(J^2) \right]$$

$\rightarrow$ invariant under rotation
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$$= \exp \left[ -\frac{1}{2} \text{Tr} (J^2) \right]$$

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$N$ real eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_N \rightarrow$ strongly correlated

Spectral statistics in RMT $\Rightarrow$ statistics of $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$
Recent excitement in statistical physics & mathematics on

\[ \lambda_{\text{max}} \Rightarrow \text{the top eigenvalue of a random matrix} \]
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Typical fluctuations (small) \\
\[ \Rightarrow \text{Tracy-Widom distribution} \]
\[ \Rightarrow \text{ubiquitous} \]

[directed polymer, random permutation, growth models, KPZ equation, sequence alignment, ....]
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This talk \[ \Rightarrow \text{Atypical rare fluctuations} \Rightarrow \text{large deviation functions} \]
Recent excitements in statistical physics & mathematics on $\lambda_{\text{max}} \Rightarrow$ the top eigenvalue of a random matrix

Typical fluctuations (small) $\Rightarrow$ Tracy-Widom distribution $\Rightarrow$ ubiquitous

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This talk $\Rightarrow$ Atypical rare fluctuations $\Rightarrow$ large deviation functions $\Rightarrow$ 3-rd order phase transition
Plan:

- Top eigenvalue $\lambda_{\text{max}}$ of a Gaussian random matrix
  $\Rightarrow$ stability of a large complex system

- Probability distribution of $\lambda_{\text{max}}$ $\Rightarrow$ Coulomb gas with a wall
- Limiting distribution: Tracy-Widom
- Physics of large deviation tails: left tail (pushed Coulomb gas)
  right tail (unpushed Coulomb gas)
  $\Rightarrow$ 3rd order phase transition: Pushed $\Leftrightarrow$ Unpushed
  (Unstable) $\Leftrightarrow$ (Stable)

- Similar 3rd order phase transition in Yang-Mills gauge theory and other systems
  $\Rightarrow$ ubiquitous

- Extension to Wishart matrices
  $\Rightarrow$ Recent experiments in coupled fiber lasers system

- Summary and Generalizations
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- Top eigenvalue $\lambda_{\text{max}}$ of a \textit{Gaussian} random matrix $\Rightarrow$ \textit{stability} of a \textit{large complex system}

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- Summary and Generalizations
I. Why $\lambda_{\text{max}}$?
Will a Large Complex System be Stable?

Gardner and Ashby\(^1\) have suggested that large complex systems which are assembled (connected) at random may be expected to be stable up to a certain critical level of connectance, and then, as this increases, to suddenly become unstable. Their conclusions were based on the trend of computer studies of systems with 4, 7 and 10 variables.

Here I complement Gardner and Ashby's work with an analytical investigation of such systems in the limit when the number of variables is large. The sharp transition from stability to instability which was the essential feature of their paper is confirmed, and I go further to see how this critical transition point scales with the number of variables \(n\) in the system, and with the average connectance \(C\) and interaction magnitude \(\alpha\) between the various variables. The object is to clarify the relation between stability and complexity in ecological systems with many interacting species, and some conclusions bearing on this question are drawn from the model.

But, just as in Gardner and Ashby's work, the formal development of the problem is a general one, and thus applies to the wide range of contexts spelled out by these authors.

Specifically, consider a system with \(n\) variables (in an ecological application these are the populations of the \(n\) interacting species) which in general may obey some quite nonlinear set of first-order differential equations. The stability of the possible equilibrium or time-independent configurations of such a system may be studied by Taylor-expanding in the neighbourhood of the equilibrium point, so that the stability of the possible equilibrium is characterized by the equation

\[
dx/dt = Ax
\]

Here in an ecological context \(x\) is the \(n \times 1\) column vector of the disturbed populations \(x_j\), and the \(n \times n\) interaction matrix \(A\) has elements \(a_{jk}\) which characterize the effect of species \(k\) on species \(j\) near equilibrium\(^2,3\). A diagram of the trophic web immediately determines which \(a_{jk}\) are zero (no web link), and the type of interaction determines the sign and magnitude of \(a_{jk}\).
Linear Stability of a Large Complex (Randomly Connected) System

• Consider a stable non-interacting population of $N$ species with equilibrium density $\rho_i^*$

• Now switch on the interaction between species

$$\frac{dx_i}{dt} = -x_i + \alpha \sum_{j=1}^{N} J_{ij} x_j$$

$J_{ij}$ → ($N \times N$) random interaction matrix

$\alpha$ → interaction strength

Question: What is the probability that the system remains stable once the interaction is switched on?

(R.M. May, Nature, 238, 413, 1972)
Consider a stable non-interacting population of $N$ species with equilibrium density $\rho_i^*$

**Stable:**

$$x_i = \rho_i - \rho_i^* \rightarrow \text{small disturbed density}$$

$$\frac{dx_i}{dt} = -x_i \rightarrow \text{relaxes back to 0}$$
Consider a stable non-interacting population of $N$ species with equilibrium density $\rho^*_i$

Stable:

$$x_i = \rho_i - \rho^*_i \rightarrow \text{small disturbed density}$$

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Now switch on the interaction between species

$$dx_i/dt = -x_i + \alpha \sum_{j=1}^{N^2} J_{ij} x_j$$

$J_{ij} \rightarrow (N \times N)$ random interaction matrix

$\alpha \rightarrow \text{interaction strength}$
Consider a stable non-interacting population of $N$ species with equilibrium density $\rho_i^\star$.

**Stable:**

\[
x_i = \rho_i - \rho_i^\star \rightarrow \text{small disturbed density}
\]

\[
dx_i / dt = -x_i \rightarrow \text{relaxes back to 0}
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Now switch on the interaction between species.

\[
dx_i / dt = -x_i + \alpha \sum_{j=1}^{N} J_{ij} x_j
\]

$J_{ij} \rightarrow (N \times N)$ random interaction matrix

$\alpha \rightarrow$ interaction strength

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- Question: What is the probability that the system remains stable once the interaction is switched on?

  (R.M. May, Nature, 238, 413, 1972)
Stability Criterion

- linear stability: \( \frac{d}{dt}[x] = [\alpha J - I][x] \) (\( J \rightarrow \text{random interaction matrix} \))

Let \( \{\lambda_1, \lambda_2, \cdots, \lambda_N\} \rightarrow \) eigenvalues of the matrix

- Stable if \( \alpha \lambda_i < 1 \) for all \( i = 1, 2, \cdots, N \) \( \Rightarrow \lambda_{\text{max}} < 1 \)

\( \alpha = w \rightarrow \text{stability criterion} \)

\( \text{Prob. (the system is stable)} = \text{Prob.} [\lambda_{\text{max}} < w] = P(w, N) \)

Cumulative distribution of the top eigenvalue

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Top eigenvalue of a random matrix: Large deviations
Stability Criterion

- linear stability: $\frac{d}{dt}[x] = [\alpha J - I][x]$ ($J \rightarrow$ random interaction matrix)

Let $\{\lambda_1, \lambda_2, \cdots, \lambda_N\} \rightarrow$ eigenvalues of the matrix $J$
Stability Criterion

- linear stability: $\frac{d}{dt}[x] = [\alpha J - I][x]$ \textit{(J \rightarrow \text{random interaction matrix})}

Let $\{\lambda_1, \lambda_2, \cdots, \lambda_N\} \rightarrow$ eigenvalues of the matrix $J$

- Stable if $\alpha \lambda_i < 1$ for all $i = 1, 2, \cdots, N$

$\Rightarrow \lambda_{\max} < \frac{1}{\alpha} = w \rightarrow$ stability criterion

$w \rightarrow$ inverse interaction strength
Stability Criterion

• linear stability: \[
\frac{d}{dt}[x] = [\alpha J - I][x] \quad (J \rightarrow \text{random interaction matrix})
\]

Let \[\{\lambda_1, \lambda_2, \cdots, \lambda_N\} \rightarrow \text{eigenvalues of the matrix } J\]

• Stable if \(\alpha \lambda_i < 1\) for all \(i = 1, 2, \cdots, N\)

\[
\Rightarrow \quad \lambda_{\text{max}} < \frac{1}{\alpha} = w \quad \rightarrow \text{stability criterion}
\]

\(w \rightarrow \text{inverse interaction strength}\)

• \(\text{Prob.}(\text{the system is stable}) = \text{Prob.}[\lambda_{\text{max}} < w] = P(w, N)\)

\(\text{Cumulative distribution of the top eigenvalue}\)
Assuming that the interaction matrix $J_{ij} \rightarrow$ Real Symmetric Gaussian

$$\text{Prob.}[J_{ij}] \propto \exp \left[ - \frac{N}{2} \sum_{i,j} J_{ij}^2 \right] \propto \exp \left[ - \frac{N}{2} \text{Tr}(J^2) \right]$$
Assuming that the interaction matrix $J_{ij} \to$ Real Symmetric Gaussian

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May observed a sharp phase transition as $N \to \infty$:

- $w = \frac{1}{\alpha} > \sqrt{2} \Rightarrow$ Stable (weakly interacting)
- $w = \frac{1}{\alpha} < \sqrt{2} \Rightarrow$ Unstable (strongly interacting)

Prob.(the system is stable) = $\text{Prob.}[\lambda_{\text{max}} < w] = P(w, N)$
Finite but Large $N$:

$$\text{Prob. (the system is stable)} = \text{Prob.} [\lambda_{\text{max}} < w] = P(w, N)$$

What happens for finite but large $N$?
Finite but Large $N$:

\[ \text{Prob. (the system is stable)} = \text{Prob.}[\lambda_{\text{max}} < w] = P(w, N) \]

What happens for finite but large $N$?

- Is there any thermodynamic sense to this phase transition?
- What is the analogue of free energy?
- What is the order of this phase transition?
II. Summary of Results
For Large but Finite $N$: Summary of Results

$P(w, N) = \text{Prob.}[\lambda_{\text{max}} \leq w]$

- finite but large $N$
- width of $O(N^{-2/3})$

Crossover function: $F_1(z) \rightarrow$ Tracy-Widom (1994)

Exact tail functions: $\Phi \pm w$ (Dean & S.M., 2006, S.M. & Vergassola, 2009)

For Large but Finite $N$: Summary of Results

\[ P(w, N) = \text{Prob.}[\lambda_{\text{max}} \leq w] \]

\[ P(w, N) \sim \exp\left[-N^2 \Phi_-(w) + \ldots\right] \quad \text{for} \quad \sqrt{2} - w \sim O(1) \]

\[ P(w, N) \sim F_1\left[\sqrt{2} N^{2/3} (w - \sqrt{2})\right] \quad \text{for} \quad |w - \sqrt{2}| \sim O(N^{-2/3}) \]

\[ P(w, N) \sim 1 - \exp\left[-N \Phi_+(w) + \ldots\right] \quad \text{for} \quad w - \sqrt{2} \sim O(1) \]
For Large but Finite $N$: Summary of Results

$p(w, N) = \text{Prob}.[ \lambda_{\text{max}} \leq w ]$

finite but large $N$

$\sim \exp \left[ -N^2 \Phi_{-}(w) + \ldots \right]$ \quad \text{for} \quad \sqrt{2} - w \sim O(1)

$\sim F_1 \left[ \sqrt{2} N^{2/3} \left( w - \sqrt{2} \right) \right]$ \quad \text{for} \quad |w - \sqrt{2}| \sim O(N^{-2/3})

$\sim 1 - \exp \left[ -N \Phi_{+}(w) + \ldots \right]$ \quad \text{for} \quad w - \sqrt{2} \sim O(1)

Crossover function: $F_1(z) \rightarrow \text{Tracy-Widom (1994)}$

Exact tail functions: $\Phi_{\mp}(w)$ (Dean & S.M., 2006, S.M. & Vergassola, 2009)

Using **Coulomb gas + Saddle point** method for large $N$:

- **Left large deviation function**:
  \[
  \Phi^-(w) = \frac{1}{10^8} \left[ 36w^2 - w^4 - (15w + w^3) \sqrt{w^2 + 6} + 27(\ln(18) - 2 \ln(w + \sqrt{6} + w^2)) \right]
  \]
  where $w < \sqrt{2}$

  As $w \to \sqrt{2}$ (from left), $\Phi^-(w) \to \frac{1}{6} \sqrt{2} (\sqrt{2} - w)^{3/2}$

- **Right large deviation function**:
  \[
  \Phi^+(w) = \frac{1}{2} w \sqrt{w^2 - 2} + \ln \left[ w - \sqrt{w^2 - 2} \sqrt{2} \right]
  \]
  where $w > \sqrt{2}$

  As $w \to \sqrt{2}$ (from right), $\Phi^+(w) \to \frac{2}{7} \frac{3}{4} \left( w - \sqrt{2} \right)^{3/2}$
Using **Coulomb gas + Saddle point** method for large $N$:

- **Left** large deviation function:

\[
\Phi_-(w) = \frac{1}{108} \left[ 36w^2 - w^4 - (15w + w^3) \sqrt{w^2 + 6} \\
+ 27 \left( \ln(18) - 2 \ln(w + \sqrt{6 + w^2}) \right) \right] \quad \text{where} \quad w < \sqrt{2}
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[D. S. Dean & S.M., PRL, 97, 160201 (2006)]
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In particular, as $w \to \sqrt{2}$ (from left), $\Phi_-(w) \to \frac{1}{6\sqrt{2}} (\sqrt{2} - w)^3$

- **Right** large deviation function:

- **Left** large deviation function:

\[
\Phi_+(w) = \sqrt{w^2 - 2} + \ln \left( \frac{w - \sqrt{w^2 - 2}}{\sqrt{2}} \right) \quad \text{where} \quad w > \sqrt{2}
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[S.M. & M. Vergassola, PRL, 102, 060601 (2009)]
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\[
\Phi_-(w) = \frac{1}{108} \left[ 36w^2 - w^4 - (15w + w^3)\sqrt{w^2 + 6} \\
+ 27 \left( \ln(18) - 2 \ln(w + \sqrt{6 + w^2}) \right) \right] \quad \text{where} \quad w < \sqrt{2}
\]

[D. S. Dean & S.M., PRL, 97, 160201 (2006)]

In particular, as $w \to \sqrt{2}$ (from left), \( \Phi_-(w) \to \frac{1}{6\sqrt{2}} (\sqrt{2} - w)^3 \)

- **Right** large deviation function:

\[
\Phi_+(w) = \frac{1}{2} w \sqrt{w^2 - 2} + \ln \left[ \frac{w - \sqrt{w^2 - 2}}{\sqrt{2}} \right] \quad \text{where} \quad w > \sqrt{2}
\]

[S.M. & M. Vergassola, PRL, 102, 060601 (2009)]

As $w \to \sqrt{2}$ (from right), \( \Phi_+(w) \to \frac{2^{7/4}}{3} (w - \sqrt{2})^{3/2} \)
Large Deviation Functions

These large deviation functions $\Phi_{\pm}(w)$ have been found useful in a large variety of problems:


[Cavagna, Garrahan, Giardina 2000,... —— Glassy systems]


3-rd Order Phase Transition

\[ P(w, N) \approx \begin{cases} 
\exp \left\{ -N^2 \Phi_-(w) + \ldots \right\} & \text{for } w < \sqrt{2} \quad \text{(unstable)} \\
1 - \exp \left\{ -N \Phi_+(w) + \ldots \right\} & \text{for } w > \sqrt{2} \quad \text{(stable)}
\end{cases} \]
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\end{cases} \]

\[
\lim_{N \to \infty} -\frac{1}{N^2} \ln [P(w, N)] = \begin{cases} 
\Phi_-(w) \sim (\sqrt{2} - w)^3 & \text{as } w \to \sqrt{2}^- \\
0 & \text{as } w \to \sqrt{2}^+
\end{cases}
\]

\[ \rightarrow \text{analogue of the free energy difference} \]
3-rd Order Phase Transition

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→ analogue of the free energy difference
$3$-rd Order Phase Transition

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\end{cases} \]

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\lim_{N \to \infty} - \frac{1}{N^2} \ln [P(w, N)] = \begin{cases} 
\Phi_-(w) \sim (\sqrt{2} - w)^3 & \text{as } w \to \sqrt{2}^- \\
0 & \text{as } w \to \sqrt{2}^+ 
\end{cases}
\]

\[ \longrightarrow \text{analogue of the free energy difference} \]

\[ \text{3-rd derivative } \to \text{discontinuous} \]

Crossover: $N \to \infty, w \to \sqrt{2}$ keeping $(w - \sqrt{2}) N^{2/3}$ fixed

\[ P(w, N) \to F_1 \left[ \sqrt{2} N^{2/3} (w - \sqrt{2}) \right] \]

\[ \to \text{Tracy-Widom} \]
Large $N$ Phase Transition: Phase Diagram

\[ \frac{1}{N} \]

$\alpha = \frac{1}{w}$

**STABLE** (weakly interacting)

**UNSTABLE** (strongly interacting)

Crossover

S.N. Majumdar

Top eigenvalue of a random matrix: Large deviations
Possible third-order phase transition in the large-\(N\) lattice gauge theory

David J. Gross
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

Edward Witten
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 10 July 1979)

The large-\(N\) limit of the two-dimensional \(U(N)\) (Wilson) lattice gauge theory is explicitly evaluated for all fixed \(\lambda = g^2 N\) by steepest-descent methods. The \(\lambda\) dependence is discussed and a third-order phase transition, at \(\lambda = 2\), is discovered. The possible existence of such a weak- to strong-coupling third-order phase transition in the large-\(N\) four-dimensional lattice gauge theory is suggested, and its meaning and implications are discussed.

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\(N = \infty\) PHASE TRANSITION IN A CLASS OF EXACTLY SOLUBLE MODEL LATTICE GAUGE THEORIES*

Spenta R. Wadia
The Enrico Fermi Institute, University of Chicago, Chicago, IL 60637, USA

Received 27 March 1980

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A nice review of large-\(N\) gauge theory: M. Marino, arXiv:1206.6272
**Large $N$ Phase Transition: Phase Diagram**

U(N) lattice gauge theory in 2-d


$\frac{1}{N}$

$0$

$g_c$

coupling strength $g$

$\alpha = \frac{1}{w_1^2}$

weakly interacting ()

STABLE

strongly interacting ()

UNSTABLE
crossover

Similar 3-rd order phase transition in $U(N)$ lattice-gauge theory in 2-d

Unstable phase $\equiv$ Strong coupling phase of Yang-Mills gauge theory

Stable phase $\equiv$ Weak coupling phase of Yang-Mills gauge theory

Tracy-Widom $\Rightarrow$ crossover function in the double scaling regime (for finite but large $N$)

S.N. Majumdar

Top eigenvalue of a random matrix: Large deviations
Large $N$ Phase Transition: Phase Diagram

$U(N)$ lattice gauge theory in 2–d

WEAK \quad \overset{\text{crossover}}{\longrightarrow} \quad \text{STRONG}

coupling strength $g$ \quad \overset{\text{g}_c}{\longrightarrow}

STABLE \quad \overset{\text{crossover}}{\longrightarrow} \quad \text{UNSTABLE}

(weakly interacting) \quad \overset{\text{strongly interacting}}{\longrightarrow}

$\alpha = \frac{1}{\sqrt{2}}$

$\frac{1}{N}$

$0$

S.N. Majumdar

Top eigenvalue of a random matrix: Large deviations
Large $N$ Phase Transition: Phase Diagram

Similar 3-rd order phase transition in $U(N)$ lattice-gauge theory in 2-d

Unstable phase $\equiv$ Strong coupling phase of Yang-Mills gauge theory
Stable phase $\equiv$ Weak coupling phase of Yang-Mills gauge theory

Tracy-Widom $\Rightarrow$ crossover function in the double scaling regime
(for finite but large $N$)
III. Coulomb Gas
Gaussian Random Matrices

- \((N \times N)\) Gaussian random matrix: \(J \equiv [J_{ij}]\)
- Ensembles: Orthogonal (GOE), Unitary (GUE) or Symplectic (GSE)

S.N. Majumdar
Top eigenvalue of a random matrix: Large deviations
Gaussian Random Matrices

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- \(N\) real eigenvalues \(\{\lambda_1, \lambda_2, \ldots, \lambda_N\}\) → correlated random variables
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• Joint distribution of eigenvalues (Wigner, 1951)

\[
P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N} \exp \left[-\frac{\beta}{2} N \sum_{i=1}^{N} \lambda_i^2 \right] \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}
\]

where the Dyson index \(\beta = 1\) (GOE), \(\beta = 2\) (GUE) or \(\beta = 4\) (GSE)
Gaussian Random Matrices

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\]

where the Dyson index \(\beta = 1\) (GOE), \(\beta = 2\) (GUE) or \(\beta = 4\) (GSE)

- \(Z_N = \text{Partition Function}\)

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ \prod_i d\lambda_i \right\} \exp\left( -\frac{\beta}{2} N \sum_{i=1}^{N} \lambda_i^2 \right) \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}
\]
Coulomb Gas Interpretation

\[ Z_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_i d\lambda_i \exp \left[ -\frac{\beta}{2} \left\{ \sum_{i=1}^{N} N \lambda_i^2 - \sum_{j \neq k} \log |\lambda_j - \lambda_k| \right\} \right] \]
Coulomb Gas Interpretation

- $Z_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \{ \prod_i d\lambda_i \} \exp \left[ -\beta \frac{1}{2} \left\{ \sum_{i=1}^{N} N \lambda_i^2 - \sum_{j \neq k} \log |\lambda_j - \lambda_k| \right\} \right]$

- 2-d Coulomb gas confined to a line (Dyson) with $\beta \to$ inverse temp.

\begin{center}
\begin{tikzpicture}
\draw (-4,0) -- (4,0);
\foreach \x in {-3,-2,-1,1,2,3}
\draw (\x,0) -- (\x,-0.1);
\foreach \y in {1,2,3,4}
\draw (0,\y) -- (-0.1,\y);
\foreach \x in {-3,-2,-1,1,2,3}
\filldraw[red] (\x,0) circle (2pt);
\filldraw[red] (0,1) circle (2pt);
\filldraw[red] (0,2) circle (2pt);
\filldraw[red] (0,3) circle (2pt);
\filldraw[red] (0,4) circle (2pt);
\end{tikzpicture}
\end{center}

\text{confining parabolic potential}
Coulomb Gas Interpretation

- $Z_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_i d\lambda_i \exp \left[ -\frac{\beta}{2} \left\{ \sum_{i=1}^{N} N \lambda_i^2 - \sum_{j \neq k} \log |\lambda_j - \lambda_k| \right\} \right]$

- 2-d Coulomb gas confined to a line (Dyson) with $\beta \to$ inverse temp.

- Balance of energy $\Rightarrow N^2 \lambda^2 \sim N^2$

- Typical eigenvalue: $\lambda_{\text{typ}} \sim O(1)$ for large $N$
• Av. density of states: \( \rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle \)
Spectral Density: Wigner’s Semicircle Law

- Av. density of states: $\rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle$

- Wigner’s Semi-circle: $\rho(\lambda, N) \xrightarrow{N \to \infty} \rho(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2}$

![Graph of Wigner's Semicircle Law](image)

- $\langle \lambda_{\text{max}} \rangle = \sqrt{2}$ for large $N$.
- $\lambda_{\text{max}}$ fluctuates from one sample to another. Prob[$\lambda_{\text{max}}, N$] = ?
Spectral Density: Wigner’s Semicircle Law

- Av. density of states: \( \rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle \)

- Wigner’s Semi-circle: \( \rho(\lambda, N) \xrightarrow{N \to \infty} \rho(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2} \)

\[ \langle \lambda_{\text{max}} \rangle = \sqrt{2} \text{ for large } N. \]

\( \lambda_{\text{max}} \) fluctuates from one sample to another. \( \text{Prob}[\lambda_{\text{max}}, N] = ? \)
Tracy-Widom distribution for $\lambda_{\text{max}}$

- Cumulative distribution of $\lambda_{\text{max}}$
- $P(w, N) = \text{Prob.}[\lambda_{\text{max}} \leq w]$ for finite but large $N$
- Typical fluctuations are distributed via Tracy-Widom ('94)
\[ \langle \lambda_{\text{max}} \rangle = \sqrt{2} ; \text{ typical fluctuation: } \int_{\lambda_{\text{max}}}^{\infty} \rho(\lambda) \, d\lambda \sim 1/N \]

Using \( \rho(\lambda) \sim (\sqrt{2} - \lambda)^{1/2} \Rightarrow |\lambda_{\text{max}} - \sqrt{2}| \sim N^{-2/3} \rightarrow \text{small} \)

[Bowick & Brezin '91, Forrester '93]
Tracy-Widom distribution for $\lambda_{\text{max}}$

- $\langle \lambda_{\text{max}} \rangle = \sqrt{2}$; typical fluctuation: $\int_{\lambda_{\text{max}}}^{\infty} \rho(\lambda) \, d\lambda \sim 1/N$

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[Bowick & Brezin '91, Forrester '93]

- typical fluctuations are distributed via Tracy-Widom ('94):

- cumulative distribution: $\text{Prob}[\lambda_{\text{max}} \leq w, N] \rightarrow F_\beta \left( \sqrt{2} N^{2/3} (w - \sqrt{2}) \right)$

S.N. Majumdar

Top eigenvalue of a random matrix: Large deviations
Tracy-Widom distribution for $\lambda_{\max}$

- $\langle \lambda_{\max} \rangle = \sqrt{2}$; typical fluctuation: $\int_{\lambda_{\max}}^{\infty} \rho(\lambda) \, d\lambda \sim 1/N$
  
  Using $\rho(\lambda) \sim (\sqrt{2} - \lambda)^{1/2}$ $\Rightarrow$ $|\lambda_{\max} - \sqrt{2}| \sim N^{-2/3} \rightarrow$ small

  [Bowick & Brezin '91, Forrester '93]

- Typical fluctuations are distributed via Tracy-Widom ('94):
  - Cumulative distribution: $\text{Prob}[\lambda_{\max} \leq w, N] \rightarrow F_\beta (\sqrt{2}N^{2/3} (w - \sqrt{2}))$
  - Prob. density (pdf): $f_\beta(x) = dF_\beta(x)/dx; \quad F_\beta(x) \rightarrow \text{Painlevé-II}$
Tracy-Widom Distribution for \( \lambda_{\text{max}} \)

Probability densities \( f(x) \)

- \( \beta = 1 \)
- \( \beta = 2 \)
- \( \beta = 4 \)

Applications: Growth models, Directed polymer, Sequence Matching ......

- Asymptotics:
  \[ f_{\beta}(x) \sim \exp\left[-\frac{\beta}{24} |x|^{3}\right] \quad \text{as} \quad x \to -\infty \]
  \[ f_{\beta}(x) \sim \exp\left[-\frac{2}{3\beta} x^{3/2}\right] \quad \text{as} \quad x \to \infty \]

\( \beta \) depends explicitly on \( \beta \).

S.N. Majumdar

Top eigenvalue of a random matrix: Large deviations
• Tracy-Widom density $f_\beta(x)$ depends explicitly on $\beta$. 

![Probability densities](image.png)
Tracy-Widom Distribution for $\lambda_{\text{max}}$

- Tracy-Widom density $f_\beta(x)$ depends explicitly on $\beta$.

- Asymptotics: $f_\beta(x) \sim \exp \left[ -\frac{\beta}{24} |x|^3 \right]$ as $x \to -\infty$
  
  $\sim \exp \left[ -\frac{2\beta}{3} x^{3/2} \right]$ as $x \to \infty$

Applications: Growth models, Directed polymer, Sequence Matching ......

(Baik, Borodin, Calabrese, Comtet, Corwin, Deift, Dotsenko, Dumitriu, Edelman, Ferrari, Forrester, Johansson, Johnstone, Le doussal, Nadal, Nechaev, O'Connell, Pèché, Prähofer, Quastel, Rains, Rambeau, Rosso, Sano, Sasamoto, Schehr, Spohn, Takeuchi, Virag, ...)

S.N. Majumdar
Top eigenvalue of a random matrix: Large deviations
Tracy-Widom Distribution for $\lambda_{\text{max}}$

- Tracy-Widom density $f_\beta(x)$ depends explicitly on $\beta$.

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  $f_\beta(x) \sim \exp \left[-\frac{\beta}{24}|x|^3\right]$ as $x \to -\infty$

  $\sim \exp \left[-\frac{2\beta}{3}x^{3/2}\right]$ as $x \to \infty$

Applications: Growth models, Directed polymer, Sequence Matching .....
Probability of Large Deviations of $\lambda_{\text{max}}$:

- Tracy-Widom law
- $\rho(\lambda, N)$ describes the probability of typical (small) fluctuations of $\lambda_{\text{max}}$ around the mean $\sqrt{2}$, i.e., when $|\lambda_{\text{max}} - \sqrt{2}| \sim N^{-2/3}$.

- Q: How to describe the probability of large (atypical) fluctuations when $|\lambda_{\text{max}} - \sqrt{2}| \sim O(1)$.

- Large deviations from mean.
Tracy-Widom law \( \text{Prob}[\lambda_{\text{max}} \leq w, N] \rightarrow F_\beta \left( \sqrt{2} N^{2/3} (w - \sqrt{2}) \right) \) describes the prob. of typical (small) fluctuations of \( \sim O(N^{-2/3}) \) around the mean \( \sqrt{2} \), i.e., when \( |\lambda_{\text{max}} - \sqrt{2}| \sim N^{-2/3} \).
Tracy-Widom law \( \text{Prob}[\lambda_{\text{max}} \leq w, N] \rightarrow F_\beta (\sqrt{2} N^{2/3} (w - \sqrt{2})) \)
describes the prob. of typical (small) fluctuations of \( \sim O(N^{-2/3}) \) around the mean \( \sqrt{2} \), i.e., when \( |\lambda_{\text{max}} - \sqrt{2}| \sim N^{-2/3} \)

Q: How to describe the prob. of large (atypical) fluctuations when \( |\lambda_{\text{max}} - \sqrt{2}| \sim O(1) \rightarrow \text{Large deviations from mean} \)
Large Deviation Tails of $\lambda_{\text{max}}$

Prob. density of the top eigenvalue:

$\rho(\lambda, N) \approx \exp\left[-\beta N^{2} \Phi - (\lambda^2)\right]$ for $\sqrt{2} - \lambda \sim O(1)$

$\rho(\lambda, N) \approx N^{-2/3} F_{\beta}\left[\sqrt{2} N^{-2/3} (\lambda - \sqrt{2})\right]$ for $|\lambda - \sqrt{2}| \sim O(N^{-2/3})$

$\rho(\lambda, N) \approx \exp\left[-\beta N^{2} \Phi + (\lambda^2)\right]$ for $\lambda - \sqrt{2} \sim O(1)$

- WIGNER SEMI-CIRCLE
- TRACY-WIDOM
- LEFT LARGE DEVIATION
- RIGHT LARGE DEVIATION
Prob. density of the top eigenvalue: \( \text{Prob.}[\lambda_{\text{max}} = w, N] \) behaves as:

\[
\sim \exp\left[-\beta N^2 \Phi_-(w)\right] \quad \text{for} \quad \sqrt{2} - w \sim O(1)
\]

\[
\sim N^{2/3} f_\beta \left[\sqrt{2} N^{2/3} \left(w - \sqrt{2}\right)\right] \quad \text{for} \quad |w - \sqrt{2}| \sim O(N^{-2/3})
\]

\[
\sim \exp\left[-\beta N \Phi_+(w)\right] \quad \text{for} \quad w - \sqrt{2} \sim O(1)
\]
IV. Saddle Point Method
Distribution of $\lambda_{\text{max}}$: Saddle Point Method

\[ \text{Prob}[\lambda_{\text{max}} \leq w, N] = \text{Prob}[\lambda_1 \leq w, \lambda_2 \leq w, \ldots, \lambda_N \leq w] = \frac{Z_N(w)}{Z_N(\infty)} \]

\[ Z_N(w) = \int_{-\infty}^{w} \ldots \int_{-\infty}^{w} \left\{ \prod_i d\lambda_i \right\} \exp \left[ -\frac{\beta}{2} \left\{ N \sum_{i=1}^{N} \lambda_i^2 - \sum_{j \neq k} \log |\lambda_j - \lambda_k| \right\} \right] \]
Distribution of $\lambda_{\text{max}}$: Saddle Point Method

\[
\text{Prob}[\lambda_{\text{max}} \leq w, N] = \text{Prob}[\lambda_1 \leq w, \lambda_2 \leq w, \ldots, \lambda_N \leq w] = \frac{Z_N(w)}{Z_N(\infty)}
\]

\[
Z_N(w) = \int_{-\infty}^{w} \ldots \int_{-\infty}^{w} \left\{ \prod d\lambda_i \right\} \exp \left[ -\frac{\beta}{2} \left\{ N \sum_{i=1}^{N} \lambda_i^2 - \sum_{j \neq k} \log |\lambda_j - \lambda_k| \right\} \right]
\]
Distribution of $\lambda_{\text{max}}$: Saddle Point Method

$$\begin{align*}
\text{Prob}[\lambda_{\text{max}} \leq w, N] &= \text{Prob}[\lambda_1 \leq w, \lambda_2 \leq w, \ldots, \lambda_N \leq w] = \frac{Z_N(w)}{Z_N(\infty)} \\
Z_N(w) &= \int_{-\infty}^{w} \cdots \int_{-\infty}^{w} \left\{ \prod_i d\lambda_i \right\} \exp \left[ -\frac{\beta}{2} \left\{ N \sum_{i=1}^{N} \lambda_i^2 - \sum_{j \neq k} \log |\lambda_j - \lambda_k| \right\} \right]\end{align*}$$
Setting up the Saddle Point Method

\[ Z_N(w) \propto \int_{-\infty}^{w} \prod_i d\lambda_i \exp \left[ -\beta N^2 E (\{\lambda_i\}) \right] \]

\[ E (\{\lambda_i\}) = \frac{1}{2N} \sum_i \lambda_i^2 - \frac{1}{2N^2} \sum_{j \neq k} \log |\lambda_j - \lambda_k| \]
• Setting up the Saddle Point Method

\[ Z_N(w) \propto \int_{-\infty}^{w} \prod d\lambda_i \exp \left[ -\beta N^2 E(\{\lambda_i\}) \right] \]

\[ E(\{\lambda_i\}) = \frac{1}{2N} \sum_i \lambda_i^2 - \frac{1}{2N^2} \sum_{j\neq k} \log |\lambda_j - \lambda_k| \]

• As \( N \to \infty \) \( \to \) discrete sum \( \to \) continuous integral:

\[ E[\rho(\lambda)] = \frac{1}{2} \left[ \int_{-\infty}^{w} \lambda^2 \rho(\lambda) d\lambda - \int_{-\infty}^{w} \int_{-\infty}^{w} \ln |\lambda - \lambda'| \rho(\lambda) \rho(\lambda') d\lambda d\lambda' \right] \]
Setting up the Saddle Point Method

- \( Z_N(w) \propto \int_{-\infty}^{w} \prod_i d\lambda_i \exp [-\beta N^2 E(\{\lambda_i\})] \)

\[
E(\{\lambda_i\}) = \frac{1}{2N} \sum_i \lambda_i^2 - \frac{1}{2N^2} \sum_{j\neq k} \log |\lambda_j - \lambda_k|
\]

- As \( N \to \infty \) \( \to \) discrete sum \( \to \) continuous integral:

\[
E(\rho(\lambda)) = \frac{1}{2} \left[ \int_{-\infty}^{w} \lambda^2 \rho(\lambda) d\lambda - \int_{-\infty}^{w} \int_{-\infty}^{w} \ln |\lambda - \lambda'| \rho(\lambda) \rho(\lambda') d\lambda d\lambda' \right]
\]

where the charge density: \( \rho(\lambda) = \frac{1}{N} \sum_i \delta(\lambda - \lambda_i) \)
Setting up the Saddle Point Method

- \[ Z_N(w) \propto \int_{-\infty}^{\infty} \prod_i d\lambda_i \exp \left[ -\beta N^2 E(\{\lambda_i\}) \right] \]

- \[ E(\{\lambda_i\}) = \frac{1}{2N} \sum_i \lambda_i^2 - \frac{1}{2N^2} \sum_{j \neq k} \log |\lambda_j - \lambda_k| \]

- As \( N \to \infty \) → discrete sum → continuous integral:

\[
E[\rho(\lambda)] = \frac{1}{2} \left[ \int_{-\infty}^{\infty} \lambda^2 \rho(\lambda) \, d\lambda - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ln |\lambda - \lambda'| \rho(\lambda) \rho(\lambda') \, d\lambda \, d\lambda' \right]
\]

where the charge density: \( \rho(\lambda) = \frac{1}{N} \sum_i \delta(\lambda - \lambda_i) \)

\[ Z_N(w) \propto \int \mathcal{D}\rho(\lambda) \exp \left[ -\beta N^2 \left\{ E[\rho(\lambda)] + C \left( \int \rho(\lambda) d\lambda - 1 \right) \right\} + O(N) \right] \]
Setting up the Saddle Point Method

\[ Z_N(w) \propto \int_{-\infty}^{\infty} \prod_i d\lambda_i \exp \left[ -\beta N^2 E(\{\lambda_i\}) \right] \]

\[ E(\{\lambda_i\}) = \frac{1}{2N} \sum_i \lambda_i^2 - \frac{1}{2N^2} \sum_{j \neq k} \log |\lambda_j - \lambda_k| \]

- As \( N \to \infty \), discrete sum \( \to \) continuous integral:

\[ E[\rho(\lambda)] = \frac{1}{2} \left[ \int_{-\infty}^{\infty} \lambda^2 \rho(\lambda) d\lambda - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ln |\lambda - \lambda'| \rho(\lambda) \rho(\lambda') d\lambda d\lambda' \right] \]

where the charge density: \( \rho(\lambda) = \frac{1}{N} \sum_i \delta(\lambda - \lambda_i) \)

\[ Z_N(w) \propto \int D\rho(\lambda) \exp \left[ -\beta N^2 \left\{ E[\rho(\lambda)] + C \left( \int \rho(\lambda) d\lambda - 1 \right) \right\} + O(N) \right] \]

- for large \( N \), minimize the action \( S[\rho(\lambda)] = E[\rho(\lambda)] + C \left( \int \rho(\lambda) d\lambda - 1 \right) \)

Saddle Point Method: \( \frac{\delta S}{\delta \rho} = 0 \Rightarrow \rho_w(\lambda) \)
Setting up the Saddle Point Method

\[ Z_N(w) \propto \int_{-\infty}^{w} \prod_i d\lambda_i \exp \left[ -\beta N^2 E(\{\lambda_i\}) \right] \]

\[ E(\{\lambda_i\}) = \frac{1}{2N} \sum_i \lambda_i^2 - \frac{1}{2N^2} \sum_{j \neq k} \log |\lambda_j - \lambda_k| \]

As \( N \to \infty \), discrete sum \( \to \) continuous integral:

\[ E[\rho(\lambda)] = \frac{1}{2} \left[ \int_{-\infty}^{w} \lambda^2 \rho(\lambda) d\lambda - \int_{-\infty}^{w} \int_{-\infty}^{w} \ln |\lambda - \lambda'| \rho(\lambda) \rho(\lambda') d\lambda d\lambda' \right] \]

where the charge density: \( \rho(\lambda) = \frac{1}{N} \sum_i \delta(\lambda - \lambda_i) \)

\[ Z_N(w) \propto \int D\rho(\lambda) \exp \left[ -\beta N^2 \left\{ E[\rho(\lambda)] + C \left( \int \rho(\lambda) d\lambda - 1 \right) \right\} + O(N) \right] \]

for large \( N \), minimize the action \( S[\rho(\lambda)] = E[\rho(\lambda)] + C \left( \int \rho(\lambda) d\lambda - 1 \right) \)

Saddle Point Method: \( \frac{\delta S}{\delta \rho} = 0 \Rightarrow \rho_w(\lambda) \)

\[ \Rightarrow Z_N(w) \sim \exp \left[ -\beta N^2 S[\rho_w(\lambda)] \right] \]
Saddle Point Solution

- saddle point \( \frac{\delta S}{\delta \delta f} = 0 \Rightarrow \)

\[
\lambda^2 - 2 \int_{-\infty}^{\omega} \rho_w(\lambda') \ln |\lambda - \lambda'| d\lambda' + C = 0
\]
Saddle Point Solution

- Saddle point \( \frac{\delta S}{\delta f} = 0 \Rightarrow \)

\[
\lambda^2 - 2 \int_{-\infty}^{w} \rho_w(\lambda') \ln |\lambda - \lambda'| d\lambda' + C = 0
\]

- Taking a derivative w.r.t. \( \lambda \) gives a singular integral Eq.

\[
\lambda = \mathcal{P} \int_{-\infty}^{w} \frac{\rho_w(\lambda') d\lambda'}{\lambda - \lambda'}
\]

for \( \lambda \in [-\infty, w] \rightarrow \text{Semi-Hilbert transform} \)

\( \rightarrow \text{force balance condition} \)
Saddle Point Solution

• saddle point \( \frac{\delta S}{\delta f} = 0 \Rightarrow \)

\[
\lambda^2 - 2 \int_{-\infty}^{w} \rho_w(\lambda') \ln|\lambda - \lambda'| \, d\lambda' + C = 0
\]

• Taking a derivative w.r.t. \( \lambda \) gives a singular integral Eq.

\[
\lambda = \mathcal{P} \int_{-\infty}^{w} \frac{\rho_w(\lambda') \, d\lambda'}{\lambda - \lambda'}
\]

for \( \lambda \in [-\infty, w] \rightarrow \) Semi-Hilbert transform

\[\rightarrow \text{force balance condition}\]

• When \( w \rightarrow \infty \): solution \( \rightarrow \) Wigner semi-circle law

\[
\rho_\infty(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2}
\]
Saddle Point Solution

- saddle point \( \frac{\delta S}{\delta f} = 0 \) \( \Rightarrow \)

\[
\lambda^2 - 2 \int_{-\infty}^{w} \rho_w(\lambda') \ln |\lambda - \lambda'| d\lambda' + C = 0
\]

- Taking a derivative w.r.t. \( \lambda \) gives a singular integral Eq.

\[
\lambda = \mathcal{P} \int_{-\infty}^{w} \frac{\rho_w(\lambda')}{\lambda - \lambda'} d\lambda'
\]

for \( \lambda \in [-\infty, w] \) \( \rightarrow \) Semi-Hilbert transform

\( \rightarrow \) force balance condition

- When \( w \rightarrow \infty \): solution \( \rightarrow \) Wigner semi-circle law

\[
\rho_\infty(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2}
\]

Exact solution for all \( w \):

[D. S. Dean & S.M., PRL, 97, 160201 (2006); PRE, 77, 041108 (2008)]
• Exact solution (D. Dean and S.M., 2006, 2008):

\[ \rho_w(\lambda) = \begin{cases} 
\frac{1}{\pi} \sqrt{2 - \lambda^2} & \text{for } w \geq \sqrt{2} \\
\frac{\sqrt{\lambda + L(w)}}{2\pi \sqrt{w - \lambda}} [w + L(w) - 2\lambda] & \text{for } w < \sqrt{2}
\end{cases} \]

where \( L(w) = \frac{[2\sqrt{w^2 + 6 - w}]}{3} \)
Exact Saddle Point Solution

- Exact solution (D. Dean and S.M., 2006, 2008):

\[ \rho_w(\lambda) = \begin{cases} 
\frac{1}{\pi} \sqrt{2 - \lambda^2} & \text{for } w \geq \sqrt{2} \\
\frac{\sqrt{\lambda + L(w)}}{2\pi \sqrt{w-\lambda}} [w + L(w) - 2\lambda] & \text{for } w < \sqrt{2}
\end{cases} \]

where \( L(w) = \frac{[2\sqrt{w^2 + 6} - w]}{3} \)

charge density \( \rho_w(\lambda) \) vs. \( \lambda \) for different \( W \)

- \( W < \sqrt{2} \): pushed critical (UNSTABLE)
- \( W = \sqrt{2} \): critical
- \( W > \sqrt{2} \): unpushed (STABLE)

\( W = \sqrt{2} \) → CRITICAL POINT
• Exact solution (D. Dean and S.M., 2006, 2008):

\[ \rho_w(\lambda) = \begin{cases} 
\frac{1}{\pi} \sqrt{2 - \lambda^2} & \text{for } w \geq \sqrt{2} \\
\frac{\sqrt{\lambda + L(w)}}{2\pi \sqrt{w - \lambda}} [w + L(w) - 2\lambda] & \text{for } w < \sqrt{2}
\end{cases} \]

where \( L(w) = \frac{2\sqrt{w^2 + 6} - w}{3} \)

charge density \( \rho_w(\lambda) \) vs. \( \lambda \) for different \( W \)

\( W < \sqrt{2} \) \hspace{1cm} \( W = \sqrt{2} \) \hspace{1cm} \( W > \sqrt{2} \)

pushed (UNSTABLE) \hspace{1cm} critical \hspace{1cm} unpushed (STABLE)

\( W = \sqrt{2} \) \hspace{1cm} CRITICAL POINT
\[
\text{Prob}[\lambda_{\text{max}} \leq w, N] = \frac{Z_N(w)}{Z_N(\infty)} \sim \exp \left[ -\beta N^2 \left\{ S[\rho_w(\lambda)] - S[\rho_\infty(\lambda)] \right\} \right] \\
\sim \exp \left[ -\beta N^2 \Phi_-(w) \right]
\]
Left Large Deviation Function

\[
\text{Prob}[\lambda_{\text{max}} \leq w, N] = \frac{Z_N(w)}{Z_N(\infty)} \sim \exp \left[ -\beta N^2 \left\{ S[\rho_w(\lambda)] - S[\rho_\infty(\lambda)] \right\} \right]
\]

\[
\sim \exp \left[ -\beta N^2 \Phi_-(w) \right]
\]

\[
\lim_{N \to \infty} -\frac{1}{N^2} \ln [P(w, N)] = \Phi_-(w) \rightarrow \text{left large deviation function}
\]

physically \( \Phi_-(w) \rightarrow \text{energy cost in pushing the Coulomb gas} \)
Left Large Deviation Function

\[
\text{Prob}[\lambda_{\text{max}} \leq w, N] = \frac{Z_N(w)}{Z_N(\infty)} \sim \exp \left[-\beta N^2 \{S[\rho_w(\lambda)] - S[\rho_\infty(\lambda)]\}\right]
\]

\[
\sim \exp \left[-\beta N^2 \Phi_-(w)\right]
\]

\[
\lim_{N \to \infty} -\frac{1}{N^2} \ln [P(w, N)] = \Phi_-(w) \rightarrow \text{left large deviation function}
\]

physically \(\Phi_-(w) \rightarrow \text{energy cost in pushing the Coulomb gas}\)

\[
\Phi_-(w) = \frac{1}{108} \left[36w^2 - w^4 - (15w + w^3)\sqrt{w^2 + 6} + 27 \left(\ln(18) - 2\ln(w + \sqrt{6 + w^2})\right)\right] \quad \text{for } w < \sqrt{2}
\]

(Dean & S.M., 2006, 2008)
Left Large Deviation Function

\[
\text{Prob}[\lambda_{\max} \leq w, N] = \frac{Z_N(w)}{Z_N(\infty)} \sim \exp \left[ -\beta N^2 \{ S[\rho_w(\lambda)] - S[\rho_\infty(\lambda)] \} \right] \\
\quad \sim \exp \left[ -\beta N^2 \Phi_-(w) \right]
\]

\[
\lim_{N \to \infty} -\frac{1}{N^2} \ln [P(w, N)] = \Phi_-(w) \to \text{left large deviation function}
\]

physically \( \Phi_-(w) \longrightarrow \text{energy cost in pushing the Coulomb gas} \)

\[
\Phi_-(w) = \frac{1}{108} \left[ 36w^2 - w^4 - (15w + w^3)\sqrt{w^2 + 6} \right. \\
\quad \left. + 27 \left( \ln(18) - 2 \ln(w + \sqrt{6 + w^2}) \right) \right] \quad \text{for} \quad w < \sqrt{2}
\]

(Dean & S.M., 2006, 2008)

Note also that \( \Phi_-(w) \approx \frac{1}{6\sqrt{2}}(\sqrt{2} - w)^3 \) as \( w \to \sqrt{2} \) from below
Matching with the left tail of Tracy-Widom:

As $w \to \sqrt{2}$ from below, $\Phi^{-}(w) \to (\sqrt{2} - w)^{3/6}\sqrt{2} \to$ matches with the left tail of the Tracy-Widom distribution.

$\lambda_{\text{max}} = w$, $N \sim \exp\left[-\beta N^2 \Phi^{-}(w)\right] \sim \exp\left[-\beta^2 \frac{1}{24} |x|^{3/2} \left|\lambda - \sqrt{2}\right|^{3/2}\right]$ as $x \to -\infty$.
Matching with the left tail of \textbf{Tracy-Widom}:

As \( w \to \sqrt{2} \) from below, \( \Phi_-(w) \to \frac{(\sqrt{2} - w)^3}{6\sqrt{2}} \)

\( \Rightarrow \) matches with the \textit{left} tail of the Tracy-Widom distribution

\[
\text{Prob.}[\lambda_{\text{max}} = w, N] \sim \exp \left[ -\beta N^2 \Phi_-(w) \right] \\
\sim \exp \left[ -\frac{\beta}{24} \sqrt{2} N^{2/3} (w - \sqrt{2})^3 \right]
\]
Matching with the left tail of Tracy-Widom:

As \( w \to \sqrt{2} \) from below, \( \Phi_-(w) \to \frac{(\sqrt{2} - w)^3}{6\sqrt{2}} \)

\[ \text{matches with the left tail of the Tracy-Widom distribution} \]

\[
\text{Prob.}[\lambda_{\text{max}} = w, N] \sim \exp \left[ -\beta N^2 \Phi_-(w) \right] \\
\sim \exp \left[ -\frac{\beta}{24} \sqrt{2} N^{2/3} (w - \sqrt{2})^3 \right]
\]

recovers the left tail of TW: \( f_\beta(x) \sim \exp \left[ -\frac{\beta}{24} |x|^3 \right] \) as \( x \to -\infty \)
For $w \geq \sqrt{2}$, saddle point solution of the charge density $\rho_w(\lambda)$ sticks to the semi-circle form: $\rho_{sc}(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2}$ for all $w \geq \sqrt{2}$.

$$\Rightarrow \text{Prob}[\lambda_{\text{max}} \leq w, N] = \frac{Z_N(w)}{Z_N(\infty)} \approx 1 \text{ as } N \to \infty$$
For \( w \geq \sqrt{2} \), saddle point solution of the charge density \( \rho_w(\lambda) \) sticks to the semi-circle form:

\[
\rho_{sc}(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2} \text{ for all } w \geq \sqrt{2}
\]

\[
\Rightarrow \text{Prob}[\lambda_{\text{max}} \leq w, N] = \frac{Z_{N}(w)}{Z_{N}(\infty)} \approx 1 \text{ as } N \rightarrow \infty
\]

\[
\Rightarrow \text{Need a different strategy}
\]
• For $w \geq \sqrt{2}$, saddle point solution of the charge density $\rho_w(\lambda)$ sticks to the semi-circle form: $\rho_{sc}(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2}$ for all $w \geq \sqrt{2}$

$$\Rightarrow \text{Prob}[\lambda_{\text{max}} \leq w, N] = \frac{Z_N(w)}{Z_N(\infty)} \approx 1 \text{ as } N \to \infty$$

$$\Rightarrow \text{Need a different strategy}$$

• Prob. density: $p(w, N) = \frac{d}{dw} P(w, N)$
Right Large Deviation Function: \( w > \sqrt{2} \)

- For \( w \geq \sqrt{2} \), saddle point solution of the charge density \( \rho_w(\lambda) \) sticks to the semi-circle form: \( \rho_{sc}(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2} \) for all \( w \geq \sqrt{2} \)

\[
\Rightarrow \text{Prob} [\lambda_{\text{max}} \leq w, N] = \frac{Z_N(w)}{Z_N(\infty)} \approx 1 \text{ as } N \to \infty
\]

\( \Rightarrow \) Need a different strategy

- Prob. density: \( p(w, N) = \frac{d}{dW} P(w, N) \)

\[
p(w, N) \propto e^{-\beta N w^2/2} \int_{-\infty}^{w} \cdots \int_{-\infty}^{w} e^{\beta \sum_{j=1}^{N-1} \ln|w-\lambda_j|} P_{N-1}(\lambda_1, \lambda_2, \ldots, \lambda_{N-1})
\]
Right Large Deviation Function: $w > \sqrt{2}$

- For $w \geq \sqrt{2}$, saddle point solution of the charge density $\rho_w(\lambda)$ sticks to the semi-circle form: $\rho_{sc}(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2}$ for all $w \geq \sqrt{2}$

  \[ \Rightarrow \text{Prob}[\lambda_{\text{max}} \leq w, N] = \frac{Z_N(w)}{Z_N(\infty)} \approx 1 \text{ as } N \to \infty \]

  \[ \Rightarrow \text{Need a different strategy} \]

- Prob. density: $p(w, N) = \frac{d}{dW} P(w, N)$

  \[ p(w, N) \propto e^{-\beta N w^2 / 2} \int_{-\infty}^{w} \cdots \int_{-\infty}^{w} e^{\beta \sum_{j=1}^{N-1} \ln |w - \lambda_j|} P_{N-1}(\lambda_1, \lambda_2, \ldots, \lambda_{N-1}) \]

  \[ \to (N - 1)-\text{fold integral} \]

WIGNER SEMI–CIRCLE
Pulled Coulomb gas

\[ p(w, N) \propto e^{-\beta N w^2 / 2} \left\langle e^{\beta \sum_j \ln |w - \lambda_j|} \right\rangle \]
Pulled Coulomb gas

\[ p(w, N) \propto e^{-\beta N w^2/2} \left< e^{\beta \sum_j \ln |w - \lambda_j|} \right> \]

Large \( N \) limit: \[ p(w, N) \propto \exp \left[ -\beta N \frac{w^2}{2} + \beta N \int \ln(w - \lambda) \rho_{sc}(\lambda) d\lambda \right] \]
Pulled Coulomb gas

\[ p(w, N) \propto e^{-\beta N w^2 / 2} \left< e^{\beta \sum_j \ln |w - \lambda_j|} \right> \]

Large \( N \) limit: \( p(w, N) \propto \exp \left[ -\beta N \frac{w^2}{2} + \beta N \int \ln(w - \lambda) \rho_{sc}(\lambda) \, d\lambda \right] \approx \exp[-\beta N \Phi_+(w)] \)

\( \Phi_+(w) = \Delta E(w) \)

\[ = \frac{w^2}{2} - \int_{-\sqrt{2}}^{\sqrt{2}} \ln(w - \lambda) \rho_{sc}(\lambda) \, d\lambda \]

\( \Rightarrow \) energy cost in pulling a charge out of the Wigner sea
Pulled Coulomb gas

\[ p(w, N) \propto e^{-\beta N w^2 / 2} \left\langle e^{\beta \sum_j \ln |w - \lambda_j|} \right\rangle \]

Large \( N \) limit: \( p(w, N) \propto \exp \left[ -\beta N \frac{w^2}{2} + \beta N \int \ln(w - \lambda) \rho_{sc}(\lambda) \, d\lambda \right] \)

\[ \sim \exp[-\beta N \Phi_+(w)] \]

\[ N \Phi_+(w) = \Delta E(w) \]
\[ = \frac{w^2}{2} - \int_{-\sqrt{2}}^{\sqrt{2}} \ln(w - \lambda) \rho_{sc}(\lambda) \, d\lambda \]

\( \Rightarrow \) energy cost in pulling a charge out of the Wigner sea

\[ \Phi_+(w) = \frac{1}{2} w \sqrt{w^2 - 2} + \ln \left[ \frac{w - \sqrt{w^2 - 2}}{\sqrt{2}} \right] \quad (w > \sqrt{2}) \]

[S.M. & Vergassola, PRL, 102, 160201 (2009)]
Matching with the right tail of Tracy-Widom

As \( w \to \sqrt{2} \) from above, \( \Phi^+ (w) \to \frac{2}{3} (w - \sqrt{2})^\frac{3}{2} \)

matches with the right tail of the Tracy-Widom distribution.

\[ \text{Prob} \left[ \lambda_{\text{max}} = w, N \right] \sim \exp \left[ -\beta N \Phi^+ (w) \right] \sim \exp \left[ -2\beta \frac{3}{3} \left| w - \sqrt{2} \right|^\frac{3}{2} \right] \]

⇒ recovers the right tail of TW:

\[ f_\beta (x) \sim \exp \left[ -2\beta \frac{3}{3} |x|^\frac{3}{2} \right] \text{ as } x \to \infty \]
As $w \to \sqrt{2}$ from above, $\Phi_+(w) \to \frac{2^{7/4}}{3}(w - \sqrt{2})^{3/2}$

$\to$ matches with the right tail of the Tracy-Widom distribution

$$\text{Prob.}[\lambda_{\max} = w, N] \sim \exp \left[ -\beta N \Phi_+(w) \right]$$

$$\sim \exp \left[ -\frac{2\beta}{3} \sqrt{2} N^{2/3} (w - \sqrt{2})^{3/2} \right]$$
As \( w \to \sqrt{2} \) from above, \( \Phi_+(w) \to \frac{2^{7/4}}{3} (w - \sqrt{2})^{3/2} \)

→ matches with the right tail of the Tracy-Widom distribution

\[
\operatorname{Prob}[\lambda_{\text{max}} = w, N] \sim \exp \left[ -\beta N \Phi_+(w) \right] \\
\sim \exp \left[ -\frac{2\beta}{3} \left| \sqrt{2} N^{2/3} (w - \sqrt{2}) \right|^{3/2} \right]
\]

⇒ recovers the right tail of TW: \( f_\beta(x) \sim \exp\left[ -\frac{2\beta}{3} |x|^{3/2} \right] \) as \( x \to \infty \)
Comparison with Simulations:

\( N \times N \) real Gaussian matrix \((\beta = 1)\): \( N = 10 \)

squares \( \rightarrow \) simulation points

red line \( \rightarrow \) Tracy-Widom

blue line \( \rightarrow \) left large deviation function \((\times N^2)\)

green line \( \rightarrow \) right large deviation function \((\times N)\).
Summary and Generalizations

Top eigenvalue of a random matrix: Large deviations

Prob. density of the top eigenvalue: $\rho(\lambda, N)$ behaves as:

$\sim \exp \left[ -\beta N^2 \Phi(\lambda) \right]$ for $\lambda \sim O(1)$

$\sim N^{2/3} f_\beta \left( \sqrt{2} \frac{N}{3} (\lambda - \sqrt{2}) \right)$ for $|\lambda - \sqrt{2}| \sim O(1)$

$\sim \exp \left[ -\beta N^2 \Phi(\lambda) + \Phi(\lambda) \right]$ for $\lambda - \sqrt{2} \sim O(1)$
Prob. density of the top eigenvalue: \( \text{Prob.} [\lambda_{\text{max}} = w, N] \) behaves as:

\[
\sim \exp \left[ -\beta N^2 \Phi_-(w) \right] \quad \text{for} \quad \sqrt{2} - w \sim O(1)
\]

\[
\sim N^{2/3} f_\beta \left[ \sqrt{2} N^{2/3} \left( w - \sqrt{2} \right) \right] \quad \text{for} \quad |w - \sqrt{2}| \sim O(N^{-2/3})
\]

\[
\sim \exp \left[ -\beta N \Phi_+(w) \right] \quad \text{for} \quad w - \sqrt{2} \sim O(1)
\]
Cumulative prob. of $\lambda_{\text{max}}$:

$$P (\lambda_{\text{max}} \leq w, N) \approx \begin{cases} \exp \{-\beta N^2 \Phi_-(w) + \ldots\} & \text{for } w < \sqrt{2} \\ 1 - A \exp \{-\beta N \Phi_+(w) + \ldots\} & \text{for } w > \sqrt{2} \end{cases}$$
Cumulative prob. of $\lambda_{\text{max}}$:

$$P (\lambda_{\text{max}} \leq w, N) \approx \begin{cases} 
\exp \left\{ -\beta N^2 \Phi_-(w) + \ldots \right\} & \text{for } w < \sqrt{2} \\
1 - A \exp \left\{ -\beta N \Phi_+(w) + \ldots \right\} & \text{for } w > \sqrt{2}
\end{cases}$$

$$\lim_{N \to \infty} -\frac{1}{\beta N^2} \ln \left[ P (\lambda_{\text{max}} \leq w) \right] = \begin{cases} 
\Phi_-(w) \sim (\sqrt{2} - w)^3 & \text{as } w \to \sqrt{2}^- \\
0 & \text{as } w \to \sqrt{2}^+
\end{cases}$$

3-rd derivative $\rightarrow$ discontinuous
Cumulative prob. of $\lambda_{\text{max}}$:

$$P (\lambda_{\text{max}} \leq w, N) \approx \begin{cases} \exp \left\{ -\beta N^2 \Phi_-(w) + \ldots \right\} & \text{for } w < \sqrt{2} \\ 1 - A \exp \left\{ -\beta N \Phi_+(w) + \ldots \right\} & \text{for } w > \sqrt{2} \end{cases}$$

$$\lim_{N \to \infty} -\frac{1}{\beta N^2} \ln [P (\lambda_{\text{max}} \leq w)] = \begin{cases} \Phi_-(w) \sim (\sqrt{2} - w)^3 & \text{as } w \to \sqrt{2}^- \\ 0 & \text{as } w \to \sqrt{2}^+ \end{cases}$$

3-rd derivative $\to$ discontinuous

- Left $\to$ strong-coupling phase $\to$ perturbative higher order corrections ($1/N$ expansion) to free energy
  
  [Borot, Eynard, S.M., & Nadal 2011]

- Right $\to$ weak-coupling phase $\to$ non-perturbative higher order corrections
  
3-rd order transition $\rightarrow$ ubiquitous

- $\lambda_{\text{max}}$ for other matrix ensembles: Wishart: $W = X^\dagger X \rightarrow (N \times N)$ $\rightarrow$ covariance matrix

Typical: Tracy-Widom

Large deviations: Exact rate functions

[Vivo, S.M., Bohigas 2007, S.M. & Vergassola 2009]
• $\lambda_{\text{max}}$ for other matrix ensembles: Wishart: $W = X^\dagger X \rightarrow (N \times N)$
  $\rightarrow$ covariance matrix

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Large deviations: Exact rate functions

[Vivo, S.M., Bohigas 2007, S.M. & Vergassola 2009]

• large $N$ gauge theory in 2-d
  [Gross, Witten, Wadia ’80, Douglas & Kazakov ’93]

• Distribution of MIMO capacity
  [Kazakopoulos et. al. 2010]

• Complexity in spin glass models
  [Fyodorov & Nadal 2013]
• $\lambda_{\text{max}}$ for other matrix ensembles: Wishart: $W = X^\dagger X \rightarrow (N \times N) \rightarrow \text{covariance matrix}$


Large deviations: Exact rate functions

[Vivo, S.M., Bohigas 2007, S.M. & Vergassola 2009]

• large $N$ gauge theory in 2-d [Gross, Witten, Wadia ’80, Douglas & Kazakov ’93]

• Distribution of MIMO capacity [Kazakopoulos et. al. 2010]

• Complexity in spin glass models [Fyodorov & Nadal 2013]

• Conductance and Shot Noise in Mesoscopic Cavities

• Entanglement entropy of a random pure state in a bipartite system

• Maximum displacement in Vicious walker problem

• Distribution of Wigner time-delay . . .

Gap between the soft edge (square-root singularity) of the Coulomb droplet and the hard wall vanishes as a control parameter $g$ goes through a critical value $g_c$:

$$\text{gap} \rightarrow 0 \ \text{as} \ g \rightarrow g_c$$
Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers

Moti Fridman, Rami Pugatch, Micha Nixon, Asher A. Friesem, and Nir Davidson
Weizmann Institute of Science, Dept. of Physics of Complex Systems, Rehovot 76100, Israel
(Dated: May 30, 2011)

We determined the probability distribution of the combined output power from twenty five coupled fiber lasers and show that it agrees well with the Tracy-Widom, Majumdar-Vergassola and Vivo-Majumdar-Bohigas distributions of the largest eigenvalue of Wishart random matrices with no fitting parameters. This was achieved with 500,000 measurements of the combined output power from the fiber lasers, that continuously changes with variations of the fiber lasers lengths. We show experimentally that for small deviations of the combined output power over its mean value the Tracy-Widom distribution is correct, while for large deviations the Majumdar-Vergassola and Vivo-Majumdar-Bohigas distributions are correct.
combined output power from fiber lasers $\propto \lambda_{\text{max}}$

$\lambda_{\text{max}} \rightarrow$ top eigenvalue of the Wishart matrix $W = X^t X$

where $X \rightarrow$ real symmetric Gaussian matrix ($\beta = 1$)
Experimental Verification with Coupled Lasers

Top eigenvalue of a random matrix: Large deviations
Experimental Verification with coupled lasers

S.N. Majumdar

Top eigenvalue of a random matrix: Large deviations
Tracy-Widom density with $\beta = 1$

Fridman et. al. arXiv:1012.1282
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• A. Scardichio (ICTP, Trieste, Italy)
• S. Tomsovic (Washington State Univ., USA)
• M. Vergassola (Institut Pasteur, Paris, France)
Selected References;

Recent review: S.M. & G. Schehr, arXiv: 1311.0580

The scaling function $F_\beta(x)$ has the expression:

- $\beta = 1$: $F_1(x) = \exp \left[ -\frac{1}{2} \int_x^\infty [(y - x)q^2(y) + q(y)] \, dy \right]$

- $\beta = 2$: $F_2(x) = \exp \left[ -\int_x^\infty (y - x)q^2(y) \, dy \right]$

- $\beta = 4$: $F_4(x) = \exp \left[ -\frac{1}{2} \int_x^\infty (y - x)q^2(y) \, dy \right] \cosh \left[ \frac{1}{2} \int_x^\infty q(y) \, dy \right]$

\[
\frac{d^2q}{dy^2} = 2q^3(y) + yq(y) \quad \text{with} \quad q(y) \to \text{Ai}(y) \quad \text{as} \quad y \to \infty \to \text{Painlevé-II}
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- Probability density: $f_\beta(x) = dF_\beta(x)/dx$
On the left side: $\lambda_{\text{max}} < \sqrt{2}$
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Adapting ‘loop (Pastur) equations’ approach developed by Chekov, Eynard and collaborators:
On the left side: \( \lambda_{\text{max}} < \sqrt{2} \)

Adapting ‘loop (Pastur) equations’ approach developed by Chekov, Eynard and collaborators:

\[
-\ln[\text{Prob}(\lambda_{\text{max}} = w, N)] = \beta \Phi_-(w) N^2 + (\beta - 2)\Phi_1(w) N + \\
+ \phi_\beta \ln N + \Phi_2(\beta, w) + O(1/N)
\]

where explicit expressions for \( \Phi_1(w) \), \( \phi_\beta \) and \( \Phi_2(\beta, w) \) were obtained recently (Borot, Eynard, S.M., & Nadal, JSTAT, P11024 (2011))
Left Large Deviation: Beyond the Leading Order

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\]

where explicit expressions for \( \Phi_1(w), \phi_\beta \) and \( \Phi_2(\beta, w) \) were obtained recently \(^{(\text{Borot, Eynard, S.M., & Nadal, JSTAT, P11024 (2011)})}\)

Setting \( w = \sqrt{2} + 2^{-1/2} N^{-2/3} x \) (with \( x < 0 \)) gives the left tail \((x \to -\infty)\) estimate of the TW density for all \( \beta \)
On the left side: $\lambda_{\text{max}} < \sqrt{2}$

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• Setting $w = \sqrt{2} + 2^{-1/2} N^{-2/3} x$ (with $x < 0$) gives the left tail $(x \to -\infty)$ estimate of the TW density for all $\beta$

$$\text{Prob.}[\lambda_{\text{max}} < \sqrt{2} + 2^{-1/2} N^{-2/3} x] \to \tau_\beta |x|^2 \exp \left[ -\frac{\beta|x|^3}{24} + \frac{\sqrt{2(\beta-2)}}{6} |x|^{3/2} \right]$$
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• Setting $w = \sqrt{2} + 2^{-1/2} N^{-2/3} x$ (with $x < 0$) gives the left tail ($x \to -\infty$) estimate of the TW density for all $\beta$

$$\text{Prob. } [\lambda_{\text{max}} < \sqrt{2} + 2^{-1/2} N^{-2/3} x] \to \tau_\beta |x|^{(\beta^2 + 4 - 6\beta)/2\beta} \exp \left[-\beta \frac{|x|^3}{24} + \frac{\sqrt{2}(\beta - 2)}{6} |x|^{3/2}\right]$$

where the constant $\tau_\beta$ is $\to$
The constant $\tau_\beta$

$$\ln \tau_\beta = \left( \frac{17}{8} - \frac{25}{24} \left( \frac{\beta^2 + 4}{2\beta} \right) \right) \ln(2) - \frac{1}{4} \ln \left( \frac{\pi \beta^2}{2} \right) +$$

$$+ \frac{\beta}{2} \left( \frac{1}{12} - \zeta'(1) \right) + \frac{\gamma_E}{6\beta} +$$

$$+ \int_0^\infty dx \left[ \frac{6x \coth(x/2) - 12 - x^2}{12x^2(e^{\beta x/2} - 1)} \right]$$

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(Borot, Eynard, S.M., & Nadal, 2011)

For $\beta = 1, 2$ and $4$

$\rightarrow$ agrees with Baik, Buckingham and DiFranco (2008)
On the right side: $\lambda_{\text{max}} > \sqrt{2}$

Using an 'orthogonal polynomial' (with an upper cut-off) method (for $\beta = 2$) and adapting the techniques used by Gross and Matytsin, '94 in the context of two-dimensional Yang-Mills theory

$$\operatorname{Prob}(\lambda_{\text{max}} = w, N) \approx \frac{1}{4 \pi \sqrt{2}} e^{-2N \Phi(w) + (w)(w^2 - 2)}$$

where $\Phi(w) = \frac{1}{2} w \sqrt{w^2 - 2} + \ln [w - \sqrt{w^2 - 2} \sqrt{2}]$

(C. Nadal and S.M., JSTAT, P04001, 2011)

Close to $w \rightarrow \sqrt{2} + 2$, this gives

$$\operatorname{Prob}.[\lambda_{\text{max}} < \sqrt{2 + 2 - \frac{1}{2} - \frac{1}{3} x} \rightarrow -\frac{1}{16} \frac{1}{2} e^{-\left(\frac{4}{3}\right) x^{3/2}}$$

precise asymptotics of the right tail of TW for $\beta = 2$ (Baik, 2006)

For general $\beta$, precise right tail of TW obtained recently (Dumaz and Virag, 2011)

As a bonus, our method also provides a 'simpler' derivation of TW distribution for $\beta = 2$ (Nadal and S.M., 2011)
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$\text{Prob}(\lambda_{\text{max}} = w, N) \approx \frac{1}{2} \pi \sqrt{2} e^{-2N \Phi^+} (w) (w^2 - 2)$

where $\Phi^+ (w) = \frac{1}{2} w \sqrt{w^2 - 2} + \ln \left[ w - \sqrt{w^2 - 2} \sqrt{2} \right]$ (C. Nadal and S.M., JSTAT, P04001, 2011)

Close to $w \to \sqrt{2}^+$, this gives

$\text{Prob}. \left[ \lambda_{\text{max}} < \sqrt{2} + \frac{3}{2} \sqrt{2} - \frac{1}{2}N - \frac{2}{3}x \right] \to \frac{1}{16} \pi x^{3/2} e^{-\left(\frac{4}{3}\right)x^{3/2}}$ (Baik, 2006)

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  (Nadal and S.M., 2011)
A simple example of **large deviation tails**

- Let \( M \rightarrow \) no. of heads in \( N \) tosses of an unbiased coin
A simple example of **large deviation** tails

- Let $M \rightarrow$ no. of heads in $N$ tosses of an unbiased coin
- Clearly $P(M, N) = \binom{N}{M} 2^{-N} (M = 0, 1, \ldots, N) \rightarrow$ binomial distribution

With mean $\langle M \rangle = \frac{N}{2}$ and variance $\sigma^2 = \langle (M - \frac{N}{2})^2 \rangle = \frac{N}{4}$
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- typical fluctuations \( M - \frac{N}{2} \sim O(\sqrt{N}) \) are well described
  by the Gaussian form: \( P(M, N) \sim \exp \left[ -\frac{2}{N} (M - \frac{N}{2})^2 \right] \)
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A simple example of large deviation tails

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- Setting \( M/N = x \) and using Stirling’s formula \( N! \sim N^{N+1/2} e^{-N} \) gives

\[
P(M = Nx, N) \sim \exp \left[ -N \Phi(x) \right]
\]

where

\[
\Phi(x) = x \log(x) + (1 - x) \log(1 - x) + \log(2) \rightarrow \text{large deviation function}
\]

\[
\Phi(x) \rightarrow \text{symmetric with a minimum at } x = 1/2 \text{ and for small arguments } |x - 1/2| \ll 1, \Phi(x) \approx 2(x - 1/2)^2 \rightarrow \text{recovery the Gaussian form near the peak}
\]
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• Clearly $P(M, N) = \binom{N}{M} 2^{-N} (M = 0, 1, \ldots, N) \to$ binomial distribution

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• Setting $M/N = x$ and using Stirling’s formula $N! \sim N^{N+1/2} e^{-N}$ gives

  $P(M = Nx, N) \sim \exp[-N\Phi(x)]$ where

  $\Phi(x) = x \log(x) + (1 - x) \log(1 - x) + \log 2 \to$ large deviation function
A simple example of large deviation tails

- Let $M \rightarrow$ no. of heads in $N$ tosses of an unbiased coin
- Clearly $P(M, N) = \binom{N}{M} 2^{-N}$ ($M = 0, 1, \ldots, N$) $\rightarrow$ binomial distribution
  
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  $\Phi(x) = x \log(x) + (1 - x) \log(1 - x) + \log 2$ $\rightarrow$ large deviation function

- $\Phi(x) \rightarrow$ symmetric with a minimum at $x = 1/2$ and
  for small arguments $|x - 1/2| << 1$, $\Phi(x) \approx 2(x - 1/2)^2$
  $\rightarrow$ recovers the Gaussian form near the peak
### Covariance Matrix

Let's consider a matrix \( X \) with dimensions \( (M \times N) \) and a transpose matrix \( X^t \) with dimensions \( (N \times M) \), where:

\[
X = \begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22} \\
X_{31} & X_{32}
\end{pmatrix}
\]

\[
X^t = \begin{pmatrix}
X_{11} & X_{21} & X_{31} \\
X_{12} & X_{22} & X_{32}
\end{pmatrix}
\]

Then, the covariance matrix \( W \) of \( X \) can be computed as:

\[
W = X^t X = \begin{pmatrix}
X_{11}^2 + X_{21}^2 + X_{31}^2 & X_{11}X_{12} + X_{21}X_{22} + X_{31}X_{32} \\
X_{12}X_{11} + X_{22}X_{21} + X_{32}X_{31} & X_{12}^2 + X_{22}^2 + X_{32}^2
\end{pmatrix}
\]

This \( W \) is the \( (N \times N) \) covariance matrix of \( X \), which is unnormalized.

**S.N. Majumdar**

Top eigenvalue of a random matrix: Large deviations
Consider $N$ students and $M = 2$ subjects (phys. and math.)

$X \rightarrow (N \times 2)$ matrix and $W = X^\dagger X \rightarrow 2 \times 2$ matrix
Consider $N$ students and $M = 2$ subjects (phys. and math.)

$X \rightarrow (N \times 2)$ matrix and $W = X^t X \rightarrow 2 \times 2$ matrix

diagonalize $w = X^t X \rightarrow [\lambda_1, \lambda_2]$

If $\lambda_1 \gg \lambda_2$
strongly correlated

random
(weak correlation)
Principal Component Analysis

Consider $N$ students and $M = 2$ subjects (phys. and math.)

$X \rightarrow (N \times 2)$ matrix and $W = X^t X \rightarrow 2 \times 2$ matrix

\[
\text{diagonalize } w = x^t x \rightarrow [\lambda_1, \lambda_2]
\]

If $\lambda_1 \gg \lambda_2$

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\[
\text{random (weak correlation)}
\]

data compression via ‘Principal Component Analysis’ (PCA)

$\Rightarrow$ practical method for image compression in computer vision
Consider $N$ students and $M = 2$ subjects (phys. and math.)

$X \rightarrow (N \times 2)$ matrix and $W = X^t X \rightarrow 2 \times 2$ matrix

Diagonalize $W = X^t X \rightarrow [\lambda_1, \lambda_2]$.

If $\lambda_1 \gg \lambda_2$, strongly correlated.

If $\lambda_1 \sim \lambda_2$, random (weak correlation).

Data compression via ‘Principal Component Analysis’ (PCA)

⇒ Practical method for image compression in computer vision

Null model → random data: $X \rightarrow$ random $(M \times N)$ matrix

$\rightarrow W = X^t X \rightarrow$ random $N \times N$ matrix (Wishart, 1928)
Generalization to Wishart Matrices

- $W = X^\dagger X \rightarrow (N \times N)$ square covariance matrix (Wishart, 1928)
- Entries of $X$ Gaussian: $\Pr[X] \propto \exp \left[-\frac{\beta}{2} N \text{Tr}(X^\dagger X)\right]$
  \[ \beta = 1 \rightarrow \text{Real entries}, \quad \beta = 2 \rightarrow \text{Complex} \]
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  $\beta = 1 \rightarrow$ Real entries, $\beta = 2 \rightarrow$ Complex

- All eigenvalues of $W = X^\dagger X$ are non-negative

\[ \rho(\lambda) = \frac{1}{2\pi \sqrt{4 - \lambda^2}} \text{ for } \lambda^2 < 4 \]
Generalization to Wishart Matrices

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- All eigenvalues of $W = X^\dagger X$ are non-negative
- Average density of states for large $N$: Marcenko-Pastur (1967)

\[
\rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle \quad \xrightarrow{N \rightarrow \infty} \quad \rho(\lambda) = \frac{1}{2\pi} \sqrt{\frac{4 - \lambda}{\lambda}}
\]
Generalization to Wishart Matrices

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- All eigenvalues of $W = X^\dagger X$ are non-negative
- Average density of states for large $N$: Marcenko-Pastur (1967)
  \[ \rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle \xrightarrow{N \to \infty} \rho(\lambda) = \frac{1}{2\pi} \sqrt{\frac{4 - \lambda}{\lambda}} \]
Distribution of $\lambda_{\text{max}}$

\[ \langle \lambda_{\text{max}} \rangle = 4 \quad (\text{as } N \to \infty) \]

- Typical fluctuations: $\lambda_{\text{max}} - 4 \sim O \left( \frac{N^2}{3} \right)$ distributed via Tracy-Widom (Johansson 2000, Johnstone 2001)

- For large deviations: $\lambda_{\text{max}} - 4 \sim O(1)$

\[
P(\lambda_{\text{max}} = w, N) \approx \begin{cases} 
\exp\left\{ -\beta N^2 \Psi - (w) \right\} & \text{for } w < 4 \\
\exp\left\{ -\beta N^2 \Psi + (w) \right\} & \text{for } w > 4 
\end{cases}
\]

\[ \Psi - (w) \text{ and } \Psi + (w) \to \text{computed exactly} \]

(Vivo, S.M. & Bohigas 2007, S.M. & Vergassola 2009)
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Using **Coulomb gas + Saddle point** method for large $N$:

- **Left** large deviation function:

$$\psi_- (w) = \ln \left[ \frac{2}{w} \right] - \frac{w - 4}{8} - \frac{(w - 4)^2}{64} \quad w \leq 4$$

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  \]
  \[\text{(Vivo, S.M. and Bohigas, 2007)}\]

- **Right** large deviation function:
  \[
  \psi_+(w) = \sqrt{\frac{w(w - 4)}{4}} + \ln \left[ \frac{w - 2 - \sqrt{w(w - 4)}}{2} \right] \quad w \geq 4
  \]
  \[\text{(S.M. and Vergassola, 2009)}\]
Other Problems with 3-rd Order Phase Transitions

- Bipartite Entanglement of a Random Pure State
  - Probability distribution of entanglement entropy

- Conductance and Shot Noise in Mesoscopic Cavities
  - Random S-matrix: Distribution of Conductance and Shot Noise
    - Damle, S.M., Tripathy, & Vivo, PRL, 107, 177206 (2011)

- Non-Intersecting Brownian Motions and Random Matrices
  - relation to 2-d Yang-Mills gauge theory

S.N. Majumdar

Top eigenvalue of a random matrix: Large deviations
• Bipartite Entanglement of a Random Pure State

Probability distribution of entanglement entropy

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