AN $N$-HOMOTOPY INVARIANCE FOR $q$-ANALOG SINGULAR HOMOLOGY

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Dedicated to Professor A. Reyes,
in the occasion of his 75th birthday.

Abstract

The Homotopy axiom for singular homology is a well known result of topological algebra. From the topological scope it asserts that any two homotopical spaces have the same singular homology and any two homotopical maps induce the same morphisms in singular homology. From the algebraic point of view it says that the singular homology is a functor that cannot distinguish between homotopical objects. This can be be understood in several ways and the usual proof of the homotopy axiom for singular homology is based in the following mathematical facts:

1. The double composition of border map $\partial$ vanishes, i.e. $\partial^2 = 0$, which means that singular chains constitute a usual chain complex.

2. The cone construction [2, p.33].

3. The cross product of singular chains, and a Leibnitz rule for this product [1].

On the other hand, the theory of $N$-complexes has raised in the last years as a new homology theory with a broad field of applications in quantum physics [4]. Let $N \geq 3$ be a prime integer. A $N$-complex is a graded module $M$ whose border map $\partial$ is a graded endomorphism that vanishes in the $N$-th composition, i.e. $\partial^N = 0$. The $m$-amplitude homologies are defined for $1 \leq m \leq N - 1$; as

$$H_m(M) = \frac{\ker(\partial^m)}{\im(\partial^{N-m})}$$

Two morphisms of $N$-differential modules $M \xrightarrow{f,g} M'$ are homotopic iff there is a sequence of morphisms of modules $M \xrightarrow{K_m} M'$, for $0 \leq m \leq N - 1$, satisfying

$$\sum_{m=0}^{N-1} (\partial')^m K_m \partial^{N-m-1} = (f - g)$$

Homotopic morphisms induce the same maps in the amplitude homologies, see [3, 5].

Some examples of $N$-complexes can be given in terms of $q$-numbers. We take a complex $N$-th root of the identity, $q \in \mathbb{C}$; i.e. $q^N = 1$. We usually assume $q = \exp(2\pi i/N)$. The basic $q$-numbers are

$$[k]_q = \frac{1 - q^k}{1 - q} = 1 + q + \cdots + q^{(k-1)} \quad \forall k \in \mathbb{N}$$

Notice that $[N]_q = 0$. The $q$-factorial numbers are

$$[k]_q ! = [1]_q \cdot [2]_q \cdots [k]_q \quad 0 \leq k \leq N - 1$$

Finally, the $q$-combinatorial numbers are

$$\left[ \begin{array}{c} k \\ l \end{array} \right]_q = \frac{[k]_q !}{[l]_q ! ([k-l]_q !)} \quad \forall 0 \leq l \leq k \leq N - 1$$
A graded differential $N$-module of $q$-singular chains $\mathcal{SC}_n(X)$ is given as follows. We define $\mathcal{SC}_n(X)$ as the free module generated by the simplexes of dimension $n$ with constants on the ring $\mathbb{Z}[q]$. The border map is

$$\partial: \mathcal{SC}_n(X) \to \mathcal{SC}_{n-1}(X) \quad \partial = \sum_{i=0}^{n} q^{i} \partial_i$$

It follows that

$$\partial^k = [k]_q! \cdot \sum_{i_1 \leq \cdots \leq i_k} q^{i_1 + \cdots + i_k} \partial_{i_k} \cdots \partial_{i_1}; \quad 0 \leq k \leq N$$

Therefore, $(\mathcal{SC}_*,(X),\partial)$ is a graded $N$-complex.

The goal of this work is to show that the $q$-Analog singular homology satisfies a homotopy axiom, with respect to the algebraic homotopy of $N$-complexes. A strategy for the proof is,

1. The $N$-th composition of border map vanishes, i.e. $\partial^N = 0$; $q$-singular chains constitute a $N$-complex.
2. A generalization of the cone construction and a suitable $q$-Leibnitz rule for this operation.
3. A suitable $q$-Leibnitz rule for the cross product of singular chains.

We extend the custom cone construction to a general convex product

$$\mathcal{SC}_m(X) \times \mathcal{SC}_n(X) \xrightarrow{*} \mathcal{SC}_{m+n+1}(X)$$

which is a bilinear operation and has a nice combinatorial behavior. Among others, we show a Leibnitz rule for $q$-chains. Given $\tau \in \mathcal{SC}_m(X)$ and $\sigma \in \mathcal{SC}_n(X)$, if $mn > 0$ then

$$\partial^k(\tau \ast \sigma) = \sum_{i=0}^{k} q^{i(m+1-k+i)} \left[ \begin{array}{c} k \\ i \end{array} \right] \partial^{m-i}(\tau) \ast \partial^i(\sigma) \quad 0 \leq k \leq \min\{m,n\}$$

The above formula is not true when $k > \min\{m,n\}$. The simplest counter-example, when $m = n = 0$ and $k = 1$, shows us that $\partial(\tau \ast \sigma) = \sigma + q \tau$. In general we can describe a sort of stationary phenomena for $k$ big enough. This leads us to the so called  *tail formulae*. For instance, if $mn > 0$ then

$$\partial^k(\partial^m(\tau) \ast \sigma) = \begin{cases} 
[m+1]_q! \cdot [k]_q \partial^{m-k+1}(\sigma) + q^k \partial^m(\tau) \ast \partial^k(\sigma) & 1 \leq k \leq n \\
[m+1]_q! \cdot [n+1]_q \partial^m(\sigma) + [n+1]_q \cdot q^{n+1} \partial^m(\tau) & k = n + 1 \\
0 & \text{else}
\end{cases}$$

Thus, the naive Leibnitz conjectured by Kapranov is not true in our context. Some differente versions of the Leibnitz rule for smooth forms have been given by Dubois-Violette in terms of Young Tableaux symmetries and tensor forms [4]. The advantage of our approach is that we show these formulæ by dealing directly with singular chains, the border map and the combinatorial properties of $q$-numbers.

Finally we consider the index map

$$(\mathcal{SC}_*(X),\partial) \to (\mathbb{Z}[q],\ast)$$

that sends each $n$-simplex to $1 \in \mathbb{Z}[q]$ in the corresponding degree, for $0 \leq n \leq N - 2$; and vanishes for $n \geq N - 1$. For $X = P$ this map is a homotopy equivalence of $N$-complexes, see §2. Our main result is that

**Theorem 1.** The index map $\mathcal{SC}_*(X) \to \mathbb{Z}[q]$ induces an isomorphism in $N$-homology.

We also hope to prove in the future the following results,

**Conjecture 2.** The following facts hold,

1. *Homotopic spaces have the same $q$-Analog singular homology.*
(2) Homotopic continuous maps induce the same map in $q$-Analogue singular homology.

References