Monodromy Of Knizhnik-Zamolodchikov Equations

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Abstract

In this paper, we recall, following [l], two constructions of (families of) representations of Artin’s braid group Bn
\[ p_{\mathcal{K}}^B : \text{Bn} \to \text{Aut}_{\text{End}}(\mathcal{W}(\mathcal{H})) \]
and
\[ p_{\mathcal{K}} : \text{Bn} \to \text{Aut}_{\text{End}}(\mathcal{W}(\mathcal{H})). \]

The representation \( p_{\mathcal{K}}^B \) is obtained analytically: it is the monodromy representation of a certain flat connection called the Knizhnik-Zamolodchikov bundle. The representation \( p_{\mathcal{K}} \) is itself obtained algebraically: it is the braid group representation associated to the universal \( \mathcal{R} \)-matrix of the quantum enveloping algebra \( \mathfrak{U}_q(\mathfrak{g}) \). Both those representations will be constructed starting from a complex semisimple Lie algebra \( \mathfrak{g} \) and objects attached to \( g \). The purpose of this paper is to give some of the tools needed to understand the statement of the following theorem:

Theorem 1 (The Kohno-Drinfeld theorem) Let \( \mathfrak{g} \) be a complex semisimple Lie algebra and let \( B \) be a g-module. The monodromy representation of a certain system of differential equations with values in \( \mathfrak{U}_q(\mathfrak{g}) \), called the Knizhnik-Zamolodchikov equations, is equivalent to the braid group representation associated to the universal \( \mathcal{R} \)-matrix of the quantum enveloping algebra \( \mathfrak{U}_q(\mathfrak{g}) \).

The KZ system

1. Monodromy representations of Artin’s braid group. In order to construct one in the above section, a monodromy representation of Artin’s braid group
\[ (s) \quad B_n \rightarrow \text{Aut}_\text{End}(\mathcal{W}(\mathcal{H})) \]

and start to find a manifold \( X \) satisfying \( p_{\mathcal{K}}(X_n) = B_n \). Here, \( X_n \) will be the configuration space of all pairwise distinct points in the complex plane, up to permutation:
\[ X_n = \mathfrak{U}_q(\mathfrak{g})/\mathfrak{U}_q(\mathfrak{g})_n \]
\[ \mathfrak{U}_q(\mathfrak{g})_n \]

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\[ X_n = \mathfrak{U}_q(\mathfrak{g})/\mathfrak{U}_q(\mathfrak{g})_n \]

Recall that \( \mathfrak{U}_q(\mathfrak{g})/\mathfrak{U}_q(\mathfrak{g})_n \) is the quotient of \( \mathfrak{U}_q(\mathfrak{g}) \) by the subalgebra \( \mathfrak{U}_q(\mathfrak{g})_n \), which consists of all elements of the form \( \sum_{i=1}^n x_i \otimes 1 \), where \( x_i \in \mathfrak{g} \) are pairwise distinct.

Theorem 2 (The Kohno-Drinfeld theorem) Let \( \mathfrak{g} \) be a complex semisimple Lie algebra and let \( B \) be a \( \mathfrak{g} \)-module. The monodromy representation of a certain system of differential equations with values in \( \mathfrak{U}_q(\mathfrak{g}) \), called the Knizhnik-Zamolodchikov equations, is equivalent to the braid group representation associated to the universal \( \mathcal{R} \)-matrix of the quantum enveloping algebra \( \mathfrak{U}_q(\mathfrak{g}) \).

The KZ system

2. Construction of a KZ system. For the above construction to make sense, one needs to specify a vector space \( W \) and endomorphisms of \( W \) satisfying relations (**), as well as an action of \( \mathfrak{g} \) on \( W \) leaving the connection \( \nabla_{s_1} s_2 \) invariant. To construct such a system, the initial data will be:
- a complex semisimple Lie algebra \( \mathfrak{g} \)
- a symmetric \( \mathfrak{g} \)-equivariant 2-terner \( t \) such that \( t \in \mathfrak{g} \) is \( \mathfrak{g} \)-equivariant and induces a \( \mathfrak{g} \)-equivariant 2-terner \( \nabla_{s_1} s_2 \) on \( W \). These \( \mathfrak{g} \)-equivariant endomorphisms satisfy relations (**).

Theorem 3 (The Kohno-Drinfeld theorem) Let \( \mathfrak{g} \) be a complex semisimple Lie algebra and let \( B \) be a \( \mathfrak{g} \)-module. The monodromy representation of a certain system of differential equations with values in \( \mathfrak{U}_q(\mathfrak{g}) \), called the Knizhnik-Zamolodchikov equations, is equivalent to the braid group representation associated to the universal \( \mathcal{R} \)-matrix of the quantum enveloping algebra \( \mathfrak{U}_q(\mathfrak{g}) \).