The work described in this paper was originally inspired by Hrushovski’s discovery [7] of striking connections between amalgamation properties and definable groupoids in models of a stable first-order theory.

Amalgamation properties have already been much studied by researchers in simple theories. The Independence Theorem, or 3-amalgamation, was used to construct canonical bases for types in such theories (reference?), and in [2], Hrushovski’s group configuration theorem for stable theories was generalized to simple theories under the assumption of 4-amalgamation over sets containing models. In [9], the $n$-amalgamation hierarchy was studied systematically. See Section 1 below for a precise definition of $n$-amalgamation.

In [7], Hrushovski showed that if a stable theory fails 3-uniqueness, then there must exist a groupoid whose sets of objects and morphisms, as well as the composition of morphisms, are definable in the theory. In [4], an explicit construction of such a groupoid was given and it was shown in [5] that the group of automorphisms of each object of such a groupoid must be abelian profinite. The morphisms in the groupoid construction in [4] arise as equivalence classes of “paths”, defined in a model-theoretic way. In some sense, the groupoid construction paralleled that of the construction of a fundamental groupoid in a topological space. Thus it seemed natural to ask whether it is possible to define the notion of a homology group in model-theoretic context and, if yes, would the homology group be linked to the group described in [4, 5].

In the companion paper [6], we defined homology groups $H_n(p)$ linked to a complete type $p$ in any simple (or even rosy) theory and showed that if $T$ has $k$-amalgamation for every $k$ between 1 and $n + 2$, then $H_n(p) = 0$ for any complete type $p$. This left open the question of the converse, whether the triviality of all homology groups could imply
n-amalgamation for every n. In this paper, we make a step in this direction, showing that a failure of 4-amalgamation in a stable theory implies the nontriviality of $H_2(p)$.

We have a conjecture as to what $H_n(p)$ should be for $n > 2$ — see Question 2.3 below. However, we ran into an obstacle when trying to extend our technique to higher homology groups: while failures of 4-amalgamation in a stable theory are always witnessed by definable non-eliminable groupoids (as explained in detail below in Section 2), we do not know a corresponding fact for higher amalgamation. Conjecturally, we believe that failures of $(n+2)$-amalgamation in a stable theory with $(n+1)$-complete amalgamation should be witnessed by some sort of definable $n$-groupoids (that is, a kind of $n$-category), but we do not know how to prove this. Also note that we make use the stability of the theory $T$ to show that the group $\Gamma_2(p)$ in our main theorem is well-defined, as well as in the construction of the groupoid in Proposition 2.7.

In Section 1, we briefly recall the definition of the homology group $H_n(p)$ from [6] and the definitions of the type amalgamation properties. In Section 2, we state and prove the main result on $H_2(p)$ in stable theories (Theorem 2.1). In the final section, Section 3, we build a family of examples showing that the abstract groups which can occur as $H_2(p)$ in a stable theory are precisely the profinite abelian groups (in a sense this complements the main result of Section 2, which implies that $H_2(p)$ is necessarily profinite).

Throughout this paper, we assume that $T$ is a complete stable theory and that $T = T^{eq}$. As usual, $\mathfrak{C}$ denotes a large, sufficiently saturated model of $T$ (the “monster model”) and all elements and sets are assumed to come from $\mathfrak{C}$. For general background on simple and stable theories, the reader is referred to the book [14], which explains nonforking, imaginaries, and much more.

1. Definition of the homology groups

In this section we recall the definition of the homology groups from our other paper [6]. In the terminology of that paper, we will only use here the “set simplices” and not the “type simplices” (which yield equivalent definitions of $H_n(p)$). We will not need to use any of the results from [6], only the definitions.

If $s$ is a set, then we consider the power set $\mathcal{P}(s)$ of $s$ to be a category with a single inclusion map $\iota_{u,v} : u \to v$ between any pair of subsets $u$ and $v$ with $u \subseteq v$. A subset $X \subseteq \mathcal{P}(s)$ is called downward-closed if whenever $u \subseteq v \in X$, then $u \in X$. In this case we consider $X$ to be a
full subcategory of $\mathcal{P}(s)$. An example of a downward-closed collection that we will use below is $\mathcal{P}^{-}(s) := \mathcal{P}(s) \setminus \{s\}.$

**Definition 1.1.** Let $A$ be a small subset of $\mathfrak{C}$. By $\mathcal{C}_A$ we denote the category of all subsets (not necessarily algebraically closed) of $\mathfrak{C}$ of size no more that $\kappa_0$, where morphisms are partial elementary maps over $A$ (that is, fixing $A$ pointwise).

**Definition 1.2.** If $A = \text{acl}(A)$ is a small subset of $\mathfrak{C}$ and $p \in S(A)$, then a **closed independent set-functor in $p$** is a functor $f : X \to \mathcal{C}_A$ such that:

1. For some finite $s \subseteq \omega$, $X$ is a downward-closed subset of $\mathcal{P}(s)$;
2. For any $t \in X$, $|f(t)| \leq (|A| + \aleph_0)^{|T|}$;
3. If $u \subseteq t \in X$, then the image $f^u_t := f(t_{u,t})$ fixes $A$ pointwise;  
4. $f(\emptyset) = A$;
5. For every $i \in s$, $f(\{i\})$ (if it is defined) is $\text{acl}(A \cup a_i)$ for some realization $a_i$ of $p$; and
6. For all non-empty $u \in X$, we have that $f(u) = \text{acl}(A \cup \bigcup_{i \in u} f^u_i(\{i\}))$ and the set $\{f^u_i(\{i\}) : i \in u\}$ is $A$-independent.

**Definition 1.3.** Let $n \geq 0$ be a natural number and $p \in S(A)$. An **$n$-simplex in $p$** is a closed independent set-functor functor $f : \mathcal{P}(s) \to \mathcal{C}$ in $p$ for some set $s \subseteq \omega$ with $|s| = n + 1$. The set $s$ is called the **support of $f$**, or $\text{supp}(f)$.

Let $S_n(p)$ denote the collection of all $n$-simplices in $p$. Then put $S(p) := \bigcup_n S_n(p)$.

Let $C_n(p)$ denote the free abelian group generated by $S_n(p)$; its elements are called **$n$-chains in $p$**, or **$n$-chains in $p$**. Similarly, we define $C(p) := \bigcup_n C_n(p)$. The **support of a chain $c$** is the union of the supports of all the simplices that appear in $c$ with a nonzero coefficient.

**Definition 1.4.** If $n \geq 1$ and $0 \leq i \leq n$, then the **$i$th boundary operator** $\partial^n_i : C_n(p) \to C_{n-1}(p)$ is defined so that if $f$ is an $n$-simplex in $p$ with domain $\mathcal{P}(s)$, where $s = \{s_0, \ldots, s_n\}$ with $s_0 < \ldots < s_n$, then

$$\partial^n_i(f) = f \upharpoonright \mathcal{P}(s \setminus \{s_i\})$$

and extended linearly to a group map on all of $C_n(p)$.

If $n \geq 1$ and $0 \leq i \leq n$, then the **boundary map** $\partial_n : C_n(p) \to C_{n-1}(p)$ is defined by the rule

$$\partial_n(c) = \Sigma_{0 \leq i \leq n} (-1)^i \partial^n_i(c).$$

We write $\partial^i$ and $\partial$ for $\partial^n_i$ and $\partial_n$, respectively, if $n$ is clear from context.
Definition 1.5. The kernel of $\partial_n$ is denoted $Z_n(p)$, and its elements are called \((n-)cycles\). The image of $\partial_{n+1}$ in $C_n(p)$ is denoted $B_n(p)$. The elements of $B_n(p)$ are called \((n-)boundaries\).

It can be shown (by the usual combinatorial argument) that $B_n(p) \subseteq Z_n(p)$, or more briefly, “$\partial_n \circ \partial_{n+1} = 0$.” Therefore we can define simplicial homology groups in the type $p$:

Definition 1.6. The $n$th \((simplicial)\) homology group of $p \in S(A)$ is

$$H_n(p) := Z_n(p)/B_n(p).$$

The definition above only makes sense for $n > 0$. Since this paper is only concerned with $H_2(p)$, we refer the curious reader to [6] for a discussion of what $H_0(p)$ might mean (we propose two different definitions there, but in either case it turns out that $H_0(p)$ gives no information about the type $p$).

Finally, we define the amalgamation properties. We use the convention that $[n]$ denotes the \((n+1)\)-element set $\{0, 1, \ldots, n\}$.

Definition 1.7. Let $n \geq 1$.

1. $p \in S(A)$ has $n$-amalgamation if for any closed independent set-functor $f : \mathcal{P}^-([n-1]) \to \mathcal{C}_A$ in the type $p$, there is an $(n-1)$-simplex $g$ in $p$ such that $g \supseteq f$.

2. $p \in S(A)$ has $n$-uniqueness if for any closed independent set-functor $f : \mathcal{P}^-([n-1]) \to \mathcal{C}_A$ in $p$ and any two $(n-1)$-simplices $g_1$ and $g_2$ in $p$ extending $f$, there is a natural isomorphism $F : g_1 \to g_2$ such that $F | \text{dom}(f)$ is the identity.

3. The theory $T$ has $n$-amalgamation (or $n$-existence) just in case all of its types $p \in S(A)$ have this property.

4. A type $p \in S(A)$ or a theory $T$ has $n$-complete amalgamation or $n$-CA if it has $k$-amalgamation for every $k$ with $1 \leq k \leq n$.

It was observed in [7] that for a stable theory, $3$-uniqueness is equivalent to $4$-existence. A link between $n$-amalgamation and the homology groups was shown in [6], where the following was proved (as Corollary 3.7 (2)):

Fact 1.8. If $n \geq 3$ and $p$ has $n$-CA, then $H_{n-2}(p) = 0$.

2. Computing $H_2(p)$ (the “Hurewicz theorem”)

We assume throughout this section that $p$ is a strong type (without loss of generality, over the empty set). We will prove that the homology group $H_2(p)$ is isomorphic to a certain automorphism group $\Gamma_2(p)$ defined below. This can be thought of as an analogue of Hurewicz’s
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Theorem in algebraic topology, which says that for a path connected topological space \( X \), the first homology group \( H_1(X) \) is isomorphic to the abelianization of the homotopy group \( \pi_1(X) \). Just as there is a higher-dimensional version of Hurewicz’s theorem for \( H_n(X) \) under the hypothesis that \( X \) is \((n-1)\)-connected, we hope that there is a higher-dimensional generalization of our result under the hypothesis that the theory \( T \) has \((n+1)\)-complete amalgamation. In other words, maybe \( n\)-CA is analogous to a topological connectedness property.

Throughout this section, “\( \bar{a} \)” denotes the algebraic closure of an element \( a \) (possibly together with a fixed ambient parameter set), considered as a possibly infinite ordered tuple, but the choice of ordering will not play any important role in what follows. Implicit in the argument below is that if \( a \equiv a_0 \), then there are orderings \( \bar{a}, \bar{a}_0 \) of their algebraic closures such that \( \bar{a} \equiv \bar{a}_0 \). Moreover, \( \text{Aut}(A/B) \) denotes the group of elementary maps from \( A \) onto \( A \) fixing \( B \) pointwise.

First, suppose that \( C = \{a_i : i \in s\} \) is an independent set of realizations of the type \( p \). Pick some \( a \) realizing \( p \) such that \( a \not\equiv C \), and let

\[
\tilde{a}_s := \bar{a}_s \cap \text{dcl} \left( \bigcup_{i \in s} \bar{a}, a_s \setminus \{i\} \right).
\]

Note that since \( T \) is stable, by stationarity, the set \( \tilde{a}_s \) does not depend on the particular choice of \( a \).

Fix some integer \( n \geq 2 \), and let \( \{a_0, \ldots, a_{n-1}\} \) be an independent set of \( n+1 \) realizations of \( p \). Recall our notation that \( [k] = \{0, \ldots, k\} \), so that \( \tilde{a}_{[n-1]} = \tilde{a}_{\{0, \ldots, n-1\}} \). Let

\[
B_n = \bigcup_{0 \leq i \leq n-1} \bar{a}_{\{0, \ldots, i, \ldots, n-1\}}.
\]

Finally, we let \( \Gamma_n(p) = \text{Aut}(\tilde{a}_{[n-1]}/B_n) \).

Note that \( \tilde{a}_{[n-1]} \) is a subset of \( \text{acl}(a_0, \ldots, a_{n-1}) \), so \( \Gamma_n(p) \) is a quotient of the full automorphism group \( \text{Aut}(\bar{a}_{[n-1]}/B_n) \) (namely, the quotient of the subgroup of all automorphisms fixing \( \tilde{a}_{[n-1]} \) pointwise).

Now we can state the main result of this section:

**Theorem 2.1.** If \( T \) is stable, \( p \) is stationary, and \( (a, b) \models p^{(2)} \), then \( H_2(p) \cong \Gamma_2(p) \).

An immediate consequence of this theorem plus Fact 1.8 above is:

**Corollary 2.2.** If \( T \) is a stable theory, then \( T \) has 3-uniqueness (or equivalently, \( T \) has 4-amalgamation) if and only if for every strong type \( p \) (possibly over a set \( A \) of extra parameters), \( H_2(p) = 0 \).
Question 2.3. If $T$ is stable with $(n + 1)$-complete amalgamation, then is $H_n(p)$ isomorphic to $\Gamma_n(p)$?

2.1. Preliminaries on definable groupoids. Here we review some material from [4] and [5] on definable groupoids that we need for the proof of Theorem 2.1. We also make a minor correction to a lemma from [4] and set some notation that will be used later. Recall that we assume $T$ is stable.

We know from [4] that in a stable theory, failures of 3-uniqueness (or equivalently, of $4$-amalgamation) are linked with type-definable connected groupoids which are not retractable. (See that paper for definitions of these terms.) It turns out that the groupoid $G$ associated to such a failure of 3-uniqueness can even be assumed to have abelian “vertex groups” $\text{Mor}_G(a, a)$ (this is proved in Section 2 of [5]).

Given an $\text{acl}(\emptyset)$-definable connected groupoid $G'$ such that the groups $G'_a := \text{Mor}_{G'}(a, a)$ are all finite and abelian, we can define canonical isomorphisms between any two groups $G'_a$ and $G'_b$ via conjugation by some (any) $h \in \text{Mor}_{G'}(a, b)$. Therefore we can quotient $\bigcup_{a \in \text{Ob}(G')} G'_a$ by this system of commuting automorphisms to get a binding group $G'$, and note that $G'$ can be thought of as a subset of $\text{acl}^{eq}(\emptyset)$. In fact, even if the mentioned groupoid $G$ is only type-definable (more precisely, relatively definable due to the explanation after Claim ??), we can still associate the binding group $G$ with a subset of $\text{acl}(\emptyset)$.

Next we recall from [5] the definition of a “full symmetric witness to the failure of 3-uniqueness.” For the present paper, we modify the definition slightly so that a full symmetric witness is a tuple $W$ containing a formula $\theta$ witnessing the key property. (Later we will need to keep track of this formula).

Definition 2.4. A full symmetric witness to non-3-uniqueness (over an algebraically closed set $A$) is a tuple $(a_0, a_1, a_2, f_{01}, f_{12}, f_{02}, \theta(x, y, z))$ such that $a_0, a_1, a_2$ and $f_{01}, f_{12}, f_{02}$ are finite tuples, $(a_0, a_1, a_2)$ is independent over $A$, $\theta(x, y, z)$ is a formula over $A$, and:

1. $f_{ij} \in \overline{a}_{ij}$;
2. $f_{01} \notin \text{dcl}(\overline{a}_0 \overline{a}_1)$;
3. $a_0 a_1 f_{01} \equiv_A a_1 a_2 f_{12} \equiv_A a_0 a_2 f_{02}$;
4. $f_{01}$ is the unique realization of $\theta(x, f_{12}, f_{02})$, the element $f_{12}$ is the unique realization of $\theta(f_{01}, y, f_{02})$, and $f_{02}$ is the unique realization of $\theta(f_{01}, f_{12}, z)$; and
5. $\text{tp}(f_{01}/\overline{a}_0 \overline{a}_1)$ is isolated by $\text{tp}(f_{01}/a_0 a_1)$. 
The following (proved in [5]) is the key technical point saying that we have “enough” symmetric witnesses:

**Proposition 2.5.** If $T$ does not have 3-uniqueness, then there is a set $A$ and a full symmetric witness to non-3-uniqueness over $A$.

In fact, if $(a_0, a_1, a_2)$ is the beginning of a Morley sequence and $f$ is any element of $\overline{a_{01}} \cap \text{dcl}(\overline{a_{02}}, \overline{a_{12}})$ which is not in $\text{dcl}(\overline{a_0}, \overline{a_1})$, then there is some full symmetric witness $(a'_0, a'_1, a'_2, f', g, h, \theta)$ such that $f \in \text{dcl}(f')$ and $a_i \in \text{dcl}(a'_i) \subseteq \overline{a_i}$ for $i = 0, 1, 2$.

The next lemma states a crucial point in the construction of type-definable groupoids from witnesses to the failure of 3-uniqueness. This was not isolated as a lemma in [4], though the idea was there.

**Lemma 2.6.** If $(a_0, a_1, a_2, f_{01}, f_{12}, f_{02}, \theta(x, y, z))$ is a full symmetric witness, and if $f \equiv_{a_0, a_1} f_{01}$ and $g \equiv_{a_1, a_2} f_{12}$, then

$$(f, g, a_0, a_1, a_2) \equiv (f_{01}, f_{12}, a_0, a_1, a_2).$$

**Proof.** By clause (5) in the definition of a full symmetric witness, $(f, a_0, a_1) \equiv (f_{01}, a_0, a_1)$ and $(g, a_1, a_2) \equiv (f_{12}, a_1, a_2)$. It follows (by the stationarity of types over $a_1$) that

$$(f, g, a_0, a_1, a_2) \equiv (f_{01}, g, a_0, a_1, a_2)$$

and

$$(f_{01}, g, a_0, a_1, a_2) \equiv (f_{01}, f_{12}, a_0, a_1, a_2),$$

and the lemma follows. ⊣

Given any full symmetric witness to the failure of 3-uniqueness, we can construct from it a connected, type-definable groupoid:

**Proposition 2.7.** Let $W = (a_0, a_1, a_2, f, g, h, \theta(x, y, z))$ be a full symmetric witness (over $\emptyset$). Then from $W$ we can construct a connected groupoid $G_W$ which is type definable over $\text{acl}(\emptyset)$ and has the following properties:

1. The objects of $G_W$ are the realizations of the type $p = \text{stp}(a_1)$.
2. Let

$$SW_{a_0, a_1} := \{ f' : f' \equiv_{a_0, a_1} f \},$$

There is a bijection $f \mapsto [f]_{G_W}^{a_0, a_1}$ from $SW_{a_0, a_1}$ onto $\text{Mor}_{G_W}(a_0, a_1)$ which is definable over $(a_0, a_1)$.
3. If $f_0, f_1 \in \text{Mor}_W(a_0, a_1)$, then $f_0 \equiv_{a_0, a_1} f_1$.
4. The “vertex groups” $\text{Mor}_{G_W}(a, a)$ are finite and abelian.
Proof. We build $\mathcal{G}_W$ using a slight modification of the construction described in subsection 2.2 of [4]. The problem with the construction in that paper is that Remark 2.8 there is incorrect as stated: in general, just because $(a,b,f) \equiv (a_0,a_1,f_{01}) \equiv (b,c,g)$, it does not follow that $(a,b,c,f,g) \equiv (a_0,a_1,a_2,f_{01},f_{12})$ (if the tuples $a_i$ are not algebraically closed, $f_{01}$ may contain elements of $\text{acl}(a_0) \backslash \text{dcl}(a_0)$, and this could cause $\text{tp}(a,f,g)$ to differ from $\text{tp}(a_0,f_{01},f_{12})$). However, Lemma 2.6 and the fact that we are using a full symmetric witness eliminates this problem. In particular, if $(a,b,f) \equiv (a_0,a_1,f_{01}) \equiv (b,c,g)$, then there is a unique element “$g \circ f$” such that $\models (f,g,g \circ f)$ and $(a,c,g \circ f) \equiv (a_0,a_2,f_{02})$.

From here, everything else in the construction of the type-definable groupoid $\mathcal{G} = \mathcal{G}_W$ in [4] works. Property (1) of the proposition follows directly from the construction, and property (2) is just like Lemma 2.14 of [4]. Because of the definable bijection in (2), any two morphisms in $\text{Mor}_G(a_0,a_1)$ have the same type, yielding (3). Finally, property (4) is Corollary 2.7 of [5].

Next, here is a more detailed version Proposition 2.15 from [5], which we will use later.

**Proposition 2.8.** Suppose that $(a_0,a_1,a_2,f_{01},f_{12},f_{02},\theta)$ is a full symmetric witness, and $\mathcal{G}$ is the associated type-definable groupoid as in Proposition 2.7. If $SW$ is the set $\{f' : \text{tp}(f'/a_0,a_1) = \text{tp}(f_{01}/a_0,a_1)\}$, then there is a group isomorphism

$$\psi_G^0 : \text{Mor}_G(a_1,a_1) \to \text{Aut}(SW/a_0,a_1)$$

defined by the rule: if $g \in \text{Mor}_G(a_1,a_1)$, then $\psi_G^0(g)$ is the unique element $\sigma \in \text{Aut}(SW/a_0,a_1)$ which induces the same left action on $\text{Mor}_G(a_0,a_1)$ as left composition by $g$.

Furthermore, the inclusion map $\text{Aut}(SW/\overline{a_0},\overline{a_1}) \to \text{Aut}(SW/a_0,a_1)$ is surjective, so we actually have an isomorphism

$$\psi_G : \text{Mor}_G(a_1,a_1) \to \text{Aut}(SW/\overline{a_0},\overline{a_1}).$$

Proof. The “Furthermore ...” clause was not in Proposition 2.15 of [5], but it follows from the fact that the witness is fully symmetric: if $f'$ is any element of $SW$, then clause (5) of the definition of a symmetric witness implies that $\text{tp}(f'/\overline{a_0},\overline{a_1}) = \text{tp}(f_{01}/\overline{a_0},\overline{a_1})$, and so there is an element $\sigma \in \text{Aut}(SW/\overline{a_0},\overline{a_1})$ such that $\sigma(f_{01}) = f'$. This means that there are at least $|\text{Mor}_G(a_1,a_0)|$ different elements in $\text{Aut}(SW/\overline{a_0},\overline{a_1})$; but, by the first part of the proposition, there are only $|\text{Mor}_G(a_1,a_0)|$ elements in $\text{Aut}(SW/a_0,a_1)$. Since this is a finite set, the injective inclusion map $\text{Aut}(SW/\overline{a_0},\overline{a_1}) \to \text{Aut}(SW/a_0,a_1)$ is surjective.
2.2. **Proof of Theorem 2.1.** We assume throughout the proof that \( p \in S(\emptyset) \) and \( \text{acl}(\emptyset) = \text{dcl}(\emptyset) \) (since we can add constants for the parameters of \( p \) if necessary). It follows directly from the definitions that if \( p = \text{tp}(a) \) and \( p' = \text{tp}(a') \) where \( a \) and \( a' \) are interalgebraic, then \( H_n(p) = H_n(p') \). Therefore, by Proposition 2.5 above, we may assume that there are some \( (a_0, a_1, a_2) \) realizing \( p^{(3)} \) and a full symmetric witness \( (a_0, a_1, a_2, f_{01}, f_{12}, f_{02}, \theta(x, y, z)) \) to this failure. We pick one such witness which we fix throughout the proof. Note that we assume the \( f_{ij} \)'s to be finite tuples, and also that there may be more than one such witness (which is the interesting case). We assume that there is at least one such witness, since otherwise \( H_2(p) \) and \( \Gamma_2(p) \) are both trivial.

As already observed in [5], the symmetric witnesses in the type \( p \) form a directed system. To make this more precise, pick some \( (a_0, a_1, a_2) \) realizing \( p^{(3)} \) (which we fix for the remainder of the subsection). Now we build a directed system of full symmetric witnesses as follows:

**Claim 2.9.** There is a directed partially ordered set \( \langle I, \leq \rangle \) and an \( I \)-indexed collection of symmetric witnesses \( \langle W_i : i \in I \rangle \) such that for any \( i \) and \( j \) in \( I \):

1. \( W_i = (a_{0i}^i, a_{1i}^i, a_{2i}^i, f_{01}^i, f_{12}^i, f_{02}^i, \theta^*_i(x_i, y_i, z_i)) \) is a full symmetric witness to failure of 3-uniqueness;
2. \( a_{0i}^i, a_{1i}^i \in \text{dcl}(f_{01}^i) \);
3. if \( i \leq j \), then \( f_{01}^i \subseteq \text{dcl}(f_{01}^j) \), \( a_{0i}^i \subseteq \text{dcl}(a_{0i}^j) \subseteq \text{dcl}(a_{0j}^j) \), and \( a_{0i}^i \equiv a_{0j}^j \); and \( a_{1i}^i \equiv a_{1j}^j \);

and satisfying the maximality conditions

\[
\overline{a_{0,1}} = \text{dcl}\left( \bigcup_{i \in I} f_{01}^i \right)
\]

and

\[
\overline{a_0} = \text{dcl}\left( \bigcup_{i \in I} a_{0i}^i \right).
\]

**Proof.** We will build the partial ordering \( \langle I, \leq \rangle \) as the union of a countable chain of partial orderings \( I_0 \subseteq I_1 \subseteq \ldots \) such that for any \( i, j \in I_n \) there is a \( k \in I_{n+1} \) such that for any \( i, j \in I_n \) there is a \( k \in I_{n+1} \) such that \( i \leq k \) and \( j \leq k \). Then the partial ordering \( I = \bigcup_{n \in \omega} I_n \) will be directed.

First, let

\[
W_i^0 = \langle a_{0i}^i, a_{1i}^i, a_{2i}^i, f_{01}^i, f_{12}^i, f_{02}^i, \theta^*_i(x_i, y_i, z_i) : i \in I_0 \rangle
\]
be any collection of full symmetric witnesses large enough to satisfy the two maximality conditions in the statement of the Claim, where $I_0$ is a trivial partial ordering in which no two distinct elements are comparable. For the induction step, suppose that we have the partial ordering $I_n$ (for some $n \in \omega$) and full symmetric witnesses $(a_0^i, \ldots, \theta_i^* (x_i, y_i, z_i))$ for each $i \in I_n$. First, we can build a partial ordering $I_{n+1}$ by adding one new point immediately above every pair of points in $I_n$ and such that any two new points in $I_{n+1} \setminus I_n$ are incomparable. Then by Proposition 2.5, there are corresponding full symmetric witnesses $(a_0^i, \ldots, \theta_i^*)$ for each $i \in I_{n+1} \setminus I_n$ such that if $j$ and $k$ are less than or equal to $i$, then $f_{i01}^j, f_{i01}^k \in dcl(f_{01}^i)$ and $a_0^j, a_0^k \in dcl(a_0^i)$. Similarly, we can ensure condition (2) (that $a_0^j, a_0^k \in dcl(f_{01}^i))$ for the new symmetric witnesses.

Let $p_i = stp(a_0^i)$ and $G_i^*$ be the type-definable groupoid constructed from the full symmetric witness $W_i$ as in Proposition 2.7 above. So $\text{Ob}(G_i) = p_i(C)$ and the groups $\text{Mor}_{G_i}^*(a_0^i, a_0^i)$ are finite and abelian, and we have the corresponding finite abelian groups $G_i^*$. As explained above, can (and will) assume that the groups $G_i^*$ are subsets of $\text{acl}(\emptyset)$.

For any $i \in I$, let $SW_i$ be the set of all realizations of $tp(f_{01}^i/a_0^i, a_1^i)$ (which is a finite set). If $(a, b) \models p_i^{(2)}$, let $SW(a, b)$ be the image of $SW_i$ under an automorphism of $C$ that maps $(a_0^i, a_0^i)$ to $(a, b)$. Recall from Proposition 2.7 that we have a definable map $f \mapsto [f]_{G_i}^{a,b}$ from $SW(a, b)$ onto $\text{Mor}_{G_i}^*(a, b)$, from which we can define an inverse map $g \mapsto \langle g \rangle_{G_i}^{a,b}$ from $\text{Mor}_{G_i}^*(a, b)$ to $SW(a, b)$. For convenience, we will write these maps as “$[\cdot]_i$” and “$\langle \cdot \rangle_i$” or even just “$[, ]_i$” and “$\langle \cdot \rangle_i$” when $(a, b)$ is clear from context.

**Lemma 2.10.** There are systems of relatively $\emptyset$-definable functions 

\[ \langle \pi_{j,i} : i \leq j, j \in I \rangle \text{ and } \langle \tau_{j,i} : i \leq j, j \in I \rangle \text{ (that is, they are the intersection of an } \emptyset \text{-definable relation with the product of their domain and range) such that whenever } i \leq j, \]

\[ (1) \; \pi_{j,i} : p_j(C) \to p_i(C), \]
\[ (2) \; \pi_{j,i} : \bigcup_{(a,b)\models p_j^{(2)}} SW(a,b) \to SW(\tau_{j,i}(a), \tau_{j,i}(b)), \]
\[ (3) \; \tau_{j,i}(a_0^j) = a_0^i, \]
\[ (4) \; \tau_{j,i}(a_1^j) = a_1^i, \]
\[ (5) \; \pi_{j,i} \text{ maps } SW_j \text{ onto } SW_i; \text{ and } \]
\[ (6) \; \pi_{j,i} f_{01}^j = f_{01}^i, \]

and whenever $i \leq j \leq k$,

\[ (7) \; \tau_{j,i} \circ \tau_{k,j} = \tau_{k,i} \text{ and } \]
\[ (8) \; \pi_{j,i} \circ \pi_{k,j} = \pi_{k,i}. \]
Proof. First, the maps $\tau_{j,i}$ can be constructed satisfying (1), (3), and (4) using the facts that $a_0^j \in \text{dcl}(a_0), a_i^j \in \text{dcl}(a_i^j)$, and $a_0^j a_i^0 = a_i^j a_i^0$ (from clause (3) of Claim 2.9). Now if $i \leq j \leq k$, since $\tau_{k,i}(x) = \tau_{j,i} \circ \tau_{k,j}(x)$ is true for $x = a_k$, this holds for every $x$ in the domain of $\tau_{k,i}$ (because the domain is a complete type), and so (7) holds.

If $i \leq j$, then since $f_{01}^j \in \text{dcl}(f_{01}^j)$, we can pick a relatively definable map $\pi_{j,i}$ such that $\pi_{j,i}(f_{01}^j) = f_{01}^j$. As before, if $i \leq j \leq k$, since $\pi_{k,i}(x) = \pi_{j,i} \circ \pi_{k,j}(x)$ holds for $x = f_{01}^k$, it holds for any $x$ in any of the sets $SW(a, b)$ for $(a, b) \models p^{(2)}$, so (8) holds.

Ideally, we would like the functions $\pi_{j,i}$ and $\tau_{j,i}$ of Lemma 2.10 to induce a commuting system of functors $F_{j,i} : G_j^* \to G_i^*$, which we could use to construct and inverse limit $G$ of $\langle G_i^* : i \in I \rangle$. This is essentially what we do, and we will then show that the group $\text{Mor}_G(\tau_0^*, \tau_0^*)$ is isomorphic to both $H_2(p)$ and $\Gamma_2(p)$. However, first we need to modify the formulas $\theta_i^*$ slightly for this to be true.

The key to making all of this work is the following technical lemma.

**Lemma 2.11.** There is a family of formulas $\langle \theta_i(x, y, z_i) : i \in I \rangle$ such that

1. $W_i$ is still a full symmetric witness with $\theta_i^*(x, y, z_i)$ replaced by $\theta_i(x, y, z_i)$ and $f_{02}^i$ replaced by another element of $SW(a_0^i, a_1^i)$, and
2. whenever $i \leq j$, $f \in SW(a_0^j, a_1^j)$, $g \in SW(a_0^i, a_1^i)$, and $h \in SW(a_0^i, a_2^i)$, then

$$\models \theta_j(f, g, h) \rightarrow \theta_i(\pi_{j,i}(f), \pi_{j,i}(g), \pi_{j,i}(h)).$$

**Proof.** Recall from above that $(a_0, a_1, a_2)$ realizes $p^{(3)}$. We use Zorn’s Lemma to find a maximal subset $J \subseteq I$ and formulas $\theta_j(x, y, z_j)$ for each $j \in J$ satisfying the following properties:

3. For every $j \in J$, there are elements $f_j$, $g_j$, and $h_j$ such that $(a_0^j, a_1^j, a_2^j, f_j, g_j, h_j, \theta_j(x, y, z_j))$ is a full symmetric witness; and
4. If $j_1, \ldots, j_n \in J$ and $(a_0^{j_1}, a_1^{j_1}, a_2^{j_1}, f_{j_1}, g_{j_1}, h_{j_1}, \theta_{j_1})$ is a full symmetric witness for $s = 1, \ldots, n$, and if $f_{j_1} \equiv g_{j_1} \equiv \cdots \equiv g_{j_n}$, then $f_{j_1} \equiv g_{j_1} \equiv \cdots \equiv g_{j_n}$.

**Claim 2.12.** $J = I$.

**Proof.** Suppose towards a contradiction that there is some $k \in I \setminus J$. Let $F_i = \langle f_{0}^i \rangle$ be a (possibly infinite) tuple listing every element of $\bigcup_{j \in J} SW(a_0^i, a_1^i)$, and let $a_i^j$ (for $i \in \{0, 1, 2\}$) be a tuple listing $\{a_i^j :
that \( f \in J \), ordered the same way as \( F_J \). Pick \( f_k \in SW(a_0^k, a_1^k) \), and then pick \( G_J = \langle g^\alpha \rangle \) and \( g_k \) such that \( F_j f_k a_0^j a_1^j \equiv G_j g_k a_0^j a_1^j \). Note that \( g^\alpha \in SW(a_1^j, a_2^j) \). Next pick a tuple \( H_j = \langle h^\alpha \rangle \) such that if \( f^\alpha \in SW(a_0^\alpha, a_1^\alpha) \), then \( \models \theta_J(f^\alpha, g^\alpha, h^\alpha) \). The element \( h_j \) is well-defined because if it happens that \( f^\alpha \) is also in \( SW(a_0^\alpha, a_1^\alpha) \) for some \( j' \neq j \), and if we let \( h' \) be the unique element such that \( \models \theta_J(f^\alpha, g^\alpha, h') \), then by property (4), the fact that \( f^\alpha f^\alpha \equiv g^\alpha g^\alpha \) implies that \( f^\alpha f^\alpha \equiv h^\alpha h' \), and so \( h' = h^\alpha \).

By the assumption (4) on the set \( J \), \( F_J \equiv H_j \). Finally, pick an element \( h_k \) such that \( F_j f_k \equiv H_j h_k \). By Corollary 2.14 of \([5]\), there is a formula \( \theta_k \) such that \( (a_0^k, a_1^k, a_2^k, f_k, g_k, h_k, \theta_k) \) is a full symmetric witness.

We claim that \( J \cup \{ k \} \) with \( \theta_k \) satisfies condition (4) above, contradicting the maximality condition on the set \( J \). Indeed, suppose that \( j_1, \ldots, j_n \in J \), and the tuples

\[(a_0^{j_s}, a_1^{j_s}, a_2^{j_s}, f_{j_s}, g_{j_s}, h_{j_s}, \theta_{j_s})\]

(for \( s = 1, \ldots, n \)) and

\[(a_0^k, a_1^k, a_2^k, f_k, g_k, h_k, \theta_k)\]

are full symmetric witnesses, and that \( f_{j_1} \ldots f_{j_n} f_k \equiv g_{j_1} \ldots g_{j_n} g_k \). By the stationarity of \( \text{tp}(f_k/a_1^k) \) and \( \text{tp}(g_k/a_1^k) \), there is a \( \sigma \in \text{Aut}(\mathcal{C}/a_0^j, a_1^j, a_2^j) \) such that \( \sigma(f_k) = f_k \) and \( \sigma(g_k) = g_k \) for the \( f_k \) and \( g_k \) from the previous paragraph. By the same argument and using the fact that \( F_J \equiv G_J \), we can also assume that if \( \sigma((f_{j_1}, \ldots, f_{j_n})) = (f_{\alpha_1}, \ldots, f_{\alpha_n}) \), then \( \sigma((g_{j_1}, \ldots, g_{j_n})) = (g_{\alpha_1}, \ldots, g_{\alpha_n}) \) (that is, the two tuples \( (f_{j_1}, \ldots, f_{j_n}) \) and \( (g_{j_1}, \ldots, g_{j_n}) \) map to corresponding subtuples of \( F_J \) and \( G_J \)). It follows that \( \sigma(h_k') = h_k \) and \( \sigma(h_{j_s}) = h_{\alpha_s} \) for each \( s \) between 1 and \( n \). By our construction, \( f_{\alpha_1} \ldots f_{\alpha_n} f_k \equiv h_{\alpha_1} \ldots h_{\alpha_n} h_k \), and so by taking preimages under \( \sigma \), we get that \( f_{j_1} \ldots f_{j_n} f_k \equiv h_{j_1} \ldots h_{j_n} h_k \).

Finally, we check that condition (2) of the lemma holds for our new formulas \( \theta_i \). Suppose that \( i \leq j \), \( f \in SW(a_0^j, a_1^j) \), \( g \in SW(a_1^j, a_2^j) \), \( h \in SW(a_0^j, a_2^j) \), and \( \models \theta_j(f, g, h) \). Let \( f_0 = \pi_{j,i}(f) \), and pick \( g_0 \) such that \( f f_0 \equiv g g_0 \). Then \( g_0 = \pi_{j,i}(g) \). Finally, let \( h_0 \) be the unique element such that \( \models \theta_i(f_0, g_0, h_0) \). By condition (4) above, \( h h_0 \equiv f f_0 \), and so \( h_0 = \pi_{j,i}(h) \). Thus \( \models \theta_i(f_0, g_0, h_0) \) as desired.

For each \( i \in I \), let \( G_i \) be the type-definable groupoid obtained from the symmetric witness \( W_i \) with the modified formula \( \theta_i \) from Lemma 2.11. Once again, the groups \( \text{Morg}_{G_i}(a, a) \) are finite and abelian.
for any $a \in \text{Ob}(\mathcal{G}_j)$, so we have the corresponding finite abelian groups $G_i$ which we consider as subsets of $\text{acl}(\emptyset)$.

**Lemma 2.13.** If $i \leq j \in I$, $(a,b,c) \models p_j^{(3)}$, $f \in \text{Mor}_{\mathcal{G}_j}(a,b)$, and $g \in \text{Mor}_{\mathcal{G}_i}(b,c)$, then

$$\left[\pi_{j,i}(\langle g \circ f \rangle_j)\right]_i = \left[\pi_{j,i}(\langle g \rangle_j)\right]_i \circ \left[\pi_{j,i}(\langle f \rangle_j)\right]_i$$

(where $\circ$ denotes composition in the groupoids $\mathcal{G}_j$ and $\mathcal{G}_i$).

**Proof.** By Proposition 2.12 of [5], $\theta_j$ defines groupoid composition between generic triples of objects in $\mathcal{G}_j$, so

$$\models \theta_j(\langle f \rangle_j, \langle g \rangle_j, \langle g \circ f \rangle_j).$$

So by Lemma 2.11,

$$\models \theta_i(\pi_{j,i}(\langle f \rangle_j), \pi_{j,i}(\langle g \rangle_j), \pi_{j,i}(\langle g \circ f \rangle_j)).$$

By Proposition 2.12 again, the Lemma follows. $\square$

If $i \leq j \in I$ and $(a,b) \models p_j^{(2)}$, then because $SW(\tau_{j,i}(a), \tau_{j,i}(b)) \subseteq \text{dcl}(SW(a,b))$, we have a canonical surjective group map

$$\rho_{j,i}^{a,b} : \text{Aut}(SW(a,b)/\overline{a}, \overline{b}) \rightarrow \text{Aut}(SW(\tau_{j,i}(a), \tau_{j,i}(b))/\overline{a}, \overline{b}),$$

and these maps satisfy the coherence condition that $\rho_{k,i}^{a,b} = \rho_{j,i}^{a,b} \circ \rho_{k,j}^{a,b}$ whenever $i \leq j \leq k$. We will write “$\rho_{j,i}^{a,b}$” for $\rho_{j,i}^{a,b}$ if $(a,b)$ is clear from context.

For every $i \in I$, we also have a group isomorphism $\psi_i : \text{Mor}_{\mathcal{G}_i}(a^i_1, a^i_1) \rightarrow \text{Aut}(SW_i/\overline{a^i_0}, \overline{a^i_1})$ as in Proposition 2.8 above.

The following is similar to Claim 2.17 of [5], except that here we have expanded this to a system of groupoid maps.

**Lemma 2.14.** For every $i \leq j \in I$, we define a map $\chi_{j,i} : \mathcal{G}_j \rightarrow \mathcal{G}_i$ by the rules:

1. If $a \in \text{Ob}(\mathcal{G}_j)$, then $\chi_{j,i}(a) = \tau_{j,i}(a)$; and
2. If $f \in \text{Mor}_{\mathcal{G}_j}(a,b)$, $c \models p_j(a,b)$, and $f = g \circ h$ for some $g \in \text{Mor}_{\mathcal{G}_j}(c,b)$ and $h \in \text{Mor}_{\mathcal{G}_j}(a,c)$, then

$$\chi_{j,i}(f) = [\pi_{j,i}(\langle h \rangle_j)]_i \circ [\pi_{j,i}(\langle g \rangle_j)]_i.$$

Then the maps $\chi_{j,i}$ satisfy:

3. $\chi_{j,i}$ is a well-defined functor;
4. $\chi_{j,i}$ is full: every morphism in $\text{Mor}(\mathcal{G}_i)$ is in the image of $\chi_{j,i}$;
5. $\chi_{j,i}$ is type-definable over $\text{acl}(\emptyset)$;
(6) If \((a, b) \models p_j^{(2)}\) and \(f \in \text{Mor}_{G_j}(a, b)\), then the formula for \(\chi_{j,i}(f)\) simplifies to
\[
\chi_{j,i}(f) = [\pi_{j,i}(\langle f \rangle_j)]_i;
\]
(7) \(\chi_{k,i} = \chi_{j,i} \circ \chi_{k,j}\) whenever \(i \leq j \leq k\); and
(8) For any \(i \leq j\), the following diagram commutes:

\[
\begin{array}{ccc}
\text{Mor}_{G_j}(a_1^i, a_1^j) & \xrightarrow{\chi_{j,i}} & \text{Mor}_{G_i}(a_i^j, a_i^j) \\
\downarrow \phi_j & & \downarrow \psi_i \\
\text{Aut}(\text{SW}_{j}/\mathcal{a}_0, \mathcal{a}_1) & \xrightarrow{\rho_{j,i}} & \text{Aut}(\text{SW}_i/\mathcal{a}_0, \mathcal{a}_1)
\end{array}
\]

Proof. Suppose that \(f \in \text{Mor}_{G_j}(a, b)\). To check that \(\chi_{j,i}(f)\) is well-defined (and does not depend on the choices of \(c, g,\) and \(h\)), first note that given \(c \models p_j^{(2)}(a, b)\) and morphisms \(g, h\) as in (2), the morphism \(h\) is uniquely determined from \(f\) and \(g\), and for any other \(g' \in \text{Mor}_{G_j}(c, b)\), \(\text{tp}(f, g/a, b, c) = \text{tp}(f, g'/a, b, c)\) (by Lemma 2.6). So the choices of \(f\) and \(g\) do not matter once we have picked \(c\), and the choice of \(c\) does not matter by the stationarity of \(p_j\).

To show that \(\chi_{j,i}\) is a functor, suppose that \(a, b,\) and \(c\) realize \(p_j\), \(f \in \text{Mor}_{G_j}(a, b)\), and \(g \in \text{Mor}_{G_i}(b, c)\). To compute the images of \(f\) and \(g\), we pick \((d, e) \models p_{j}^{(2)}(a, b, c)\) and \(f_0 \in \text{Mor}_{G_j}(a, d), f_1 \in \text{Mor}_{G_j}(d, b), g_0 \in \text{Mor}_{G_j}(b, e),\) and \(g_1 \in \text{Mor}_{G_i}(e, c)\) such that \(f = f_1 \circ f_0\) and \(g = g_1 \circ g_0 \). Then by the definition given in (2) of the Lemma,

\[
\chi_{j,i}(f) = [\pi_{j,i}(\langle g_1 \circ g_0 \circ f_1 \rangle_j)]_i \circ [\pi_{j,i}(\langle f_0 \rangle_j)]_i.
\]

By Lemma 2.13 twice, this equals

\[
[\pi_{j,i}(\langle g_1 \rangle_j)]_i \circ [\pi_{j,i}(\langle g_0 \rangle_j)]_i \circ [\pi_{j,i}(\langle f_1 \rangle_j)]_i \circ [\pi_{j,i}(\langle f_0 \rangle_j)]_i.
\]

But the composition of the first two terms above equals \(\chi_{j,i}(g)\) and the composition of the third and fourth terms equals \(\chi_{j,i}(f)\), so \(\chi_{j,i}(g \circ f) = \chi_{j,i}(g) \circ \chi_{j,i}(f)\).

Suppose that \(a, b \in \text{Ob}(G_j)\) and \(f \in \text{Mor}_{G_j}(a, b)\). Pick some \(c \models p_i(a, b)\), and pick \(g \in \text{Mor}_{G_i}(c, b)\) and \(h \in \text{Mor}_{G_i}(a, c)\) such that \(f = g \circ h\). Since

\[
\langle g \rangle_i, c, b) \equiv (f_{01}^i, a_0^i, a_1^i) \equiv (\langle h \rangle_i, a, c),
\]
we can find elements \(g'\) and \(h'\) such that \(\pi_{j,i}(g') = \langle g \rangle_i\) and \(\pi_{j,i}(h') = \langle h \rangle_i\). Let \(f^* = [g']_j \circ [h']_j\). Unwinding the definitions, we see that

\[
\chi_{j,i}(f^*) = [\pi_{j,i}(g')]_i \circ [\pi_{j,i}(h')]_i = [\langle g \rangle_i]_i \circ [\langle h \rangle_i]_i = g \circ h = f.
\]

This establishes that the functor \(\chi_{j,i}\) is full.
The fact that $\chi_{j,i}$ is type-definable is simply by the definability of types in stable theories, and in fact the action of $\chi_{j,i}$ on the objects and morphisms of $\mathcal{G}_j$ is given by the intersection of a definable set with the type-definable sets $\text{Ob}(\mathcal{G}_j)$ and $\text{Mor}(\mathcal{G}_j)$.

The formula (6) follows directly from the definition of $\chi_{j,i}(f)$ in (2) and Lemma 2.13.

Next we prove (7). Suppose that $i \leq j \leq k$. If $a \in \text{Ob}(\mathcal{G}_k)$, then $\chi_{j,i} \circ \chi_{k,j}(a) = \tau_{j,i}(\tau_{k,j}(a)) = \tau_{k,i}(a) = \chi_{k,i}(a)$. If $a, b, c \in \text{Ob}(\mathcal{G}_k)$ and $f = g \circ h$ are in (2) of the Lemma (with $j$ replaced by $k$), then by the definition of the $\chi$ maps,

$$
\chi_{j,i} \circ \chi_{k,j}(f) = \chi_{j,i} \left( [\pi_{k,j}((h)_k)]_j \circ [\pi_{k,j}((g)_k)]_j \right) = [\pi_{j,i} \left( [\pi_{k,j}((h)_k)]_j \right)]_i \circ [\pi_{j,i} \left( [\pi_{k,j}((g)_k)]_j \right)]_i = [\pi_{j,i}((h)_k)]_i \circ [\pi_{k,i}((g)_k)]_i = \chi_{k,i}(f).
$$

Finally, we check (8). Suppose $i \leq j$ and $f \in \text{Mor}_{\mathcal{G}_j}(a^j_1, a^j_0)$. To show that $\psi_i(\chi_{j,i}(f)) = \rho_{j,i}(\psi_j(f))$, we pick some arbitrary $k_0 \in \text{Mor}_{\mathcal{G}_i}(a^i_0, a^i_1)$ and show that

$$
[\psi_i(\chi_{j,i}(f))]_i (k_0) = [\rho_{j,i}(\psi_j(f))]_i (k_0).
$$

On the one hand, by definition of $\psi_i$,

$$
[\psi_i(\chi_{j,i}(f))]_i (k_0) = \chi_{j,i}(f) \circ k_0.
$$

To compute the right-hand side of equation 9, pick some $k \in \text{Mor}_{\mathcal{G}_j}(a^j_0, a^j_1)$ such that $[\pi_{j,i}((k)_j)]_i = k_0$. Then

$$
[\psi_j(f)]_i (k) = f \circ k,
$$

and $\rho_{j,i}(\psi_j(f))$ must move $k_0 = [\pi_{j,i}((k)_j)]_i$ to the element which is defined from $[\psi_j(f)]_i (k)$ in the same way that $k_0$ is defined from $k$, so

$$
[\rho_{j,i}(\psi_j(f))]_i (k_0) = [\pi_{j,i}((f \circ k)_j)]_i.
$$

By (6) and the functoriality of $\chi_{j,i}$,

$$
[\rho_{j,i}(\psi_j(f))]_i (k_0) = \chi_{j,i}(f \circ k) = \chi_{j,i}(f) \circ \chi_{j,i}(k) = \chi_{j,i}(f) \circ [\pi_{j,i}((k)_j)]_i = \chi_{j,i}(f) \circ k_0.
$$

So both sides of equation 9 equal $\chi_{j,i}(f) \circ k_0$, and we are done. \(\dashleftarrow\)
Finally, we define maps on the $p$-simplices and homology groups. Throughout, we will work with the set homology group (and set-simplices, et cetera) for convenience.

First, for every $i \in I$, we pick an arbitrary “selection function” $\alpha_i^0 : S_0 C(p) \to p_i(\mathcal{C})$ such that $\alpha_i^0(a) \in dcl(a)$. (This is a technical point, but the $0$-simplices in $S_0 C(p)$ are algebraic closures of realizations of $p_i$, and there might be no canonical way to get a realization of $p_i$ from a $0$-simplex. Thus we need the choice functions $\alpha_i^0$.)

Next, we pick selection functions $\alpha_i : S_1 C(p) \to \text{Mor}(\mathcal{G}_i)$ (for every $i \in I$) as follows. Suppose that $\text{dom}(f) = \mathcal{P}(\{n_0, n_1\})$ for $n_0 < n_1$, and for $x \in \{n_0, n_1\}$, let “$f_x$” stand for $f_{\{n_0,n_1\}}(\alpha_i^0(f \upharpoonright \mathcal{P}(\{x\})))$ (remembering that things in the image of $\alpha_i^0$ are realizations of $p_i$, which are also objects in $\text{Ob}(\mathcal{G}_i)$). Then we pick $\alpha_i(f)$ such that $\alpha_i(f) \in \text{Mor}_{\mathcal{G}_i}(f_{n_0}, f_{n_1})$. Just as in the proof of Lemma 2.10, we can use an inductive argument to ensure that if $i \leq j$ then $\chi_{j,i}(\alpha_j(f)) = \alpha_i(f)$.

Finally, want to extend $\alpha_i$ to a selection function $\epsilon_i : S_2 C(p) \to G_i$. To ease notation here and in what follows, we set the following notation:

**Notation 2.15.** Whenever $f \in S_i C(p)$, $\text{dom}(f) = \mathcal{P}(s)$, and $k \in s$, let

$$f_{k,s}^i := f_{s}^{\{k\}}(\alpha_i^0(f \upharpoonright \mathcal{P}(\{k\})),$$

and note that $f_{k,s}^i$ is a realization of $p_i$, that is, an object in $\mathcal{G}_i$. Similarly, if $\{k, \ell\} \subseteq s$ and $k < \ell$, let

$$f_{\{k,\ell\},s}^i := f_{s}^{\{k,\ell\}}(\alpha_i(f \upharpoonright \mathcal{P}(\{k, \ell\})),$$

which is a morphism in $\text{Mor}_{\mathcal{G}_i}(f_{k,s}^i, f_{\ell,s}^i)$.

**Definition 2.16.** We define $\epsilon_i : S_2 C(p) \to G_i$ by the rule: if $\text{dom}(f) = \mathcal{P}(s)$, where $s = \{n_0, n_1, n_2\}$ and $n_0 < n_1 < n_2$, then we define $\epsilon_i(f)$ as

$$\epsilon_i(f) := \left[ (f_{\{n_0,n_2\},s}^i)^{-1} \circ f_{\{n_1,n_2\},s}^i \circ f_{\{n_0,n_1\},s}^i \right]_{G_i}.$$

(Recall that if $f \in \text{Mor}_{\mathcal{G}_i}(a, a)$, then “[f]$_{G_i}$” denotes the corresponding element of the group $G_i$.)

These functions $\epsilon_i$ can be extended linearly from $S_2 C(p)$ to the collection of all $2$-chains $C_2(p)$, and by abuse of notation we also call this new function $\epsilon_i$.

The next lemma is a technical point that will be useful for later computations.

**Lemma 2.17.** If $i \in I$ and $f \in S_n(p)$ for any $n \geq 3$, $\text{dom}(g) = \mathcal{P}(t)$, and $\{a, b, c\} \subseteq s \subseteq t$ with $a < b < c$, then

$$\epsilon_i(f \upharpoonright \mathcal{P}(\{a, b, c\})) = \left[ (f_{\{a,c\},s}^i)^{-1} \circ f_{\{b,c\},s}^i \circ f_{\{a,b\},s}^i \right]_{G_i}.$$
Proof. Remember that we identify the elements of \( G_i \) with elements of \( \text{acl}(\emptyset) \). Because the transition map \( f^{(a,b,c)}_s \) fixes \( \text{acl}(\emptyset) \) pointwise,
\[
f^{(a,b,c)}_s(\epsilon_i(f \upharpoonright \{a, b, c\})) = \epsilon_i(f \upharpoonright \{a, b, c\}).
\]
Therefore the left-hand side of the equation above equals \( f^{(a,b,c)}_s(\epsilon_i(f \upharpoonright \{a, b, c\})) \), which is the equivalence class (in \( G_i \)) of
\[
\left[ f^{(a,b,c)}_s \circ f^{(a,c)}_i(\alpha_i(f \upharpoonright \mathcal{P}(\{a, c\}))) \right]^{-1} \circ
\left[ f^{(a,b,c)}_s \circ f^{(a,b,c)}_i(\alpha_i(f \upharpoonright \mathcal{P}(\{b, c\}))) \circ f^{(a,b,c)}_s \circ f^{(a,b,c)}_i(\alpha_i(f \upharpoonright \mathcal{P}(\{a, b\}))) \right]^{-1} \circ
\left[ f^{(a,c)}_s(\alpha_i(f \upharpoonright \mathcal{P}(\{a, c\}))) \right]^{-1} \circ
\left[ f^{(b,c)}_s(\alpha_i(f \upharpoonright \mathcal{P}(\{b, c\}))) \circ f^{(a,b)}_s(\alpha_i(f \upharpoonright \mathcal{P}(\{a, b\}))) \right],
\]
as desired.

Lemma 2.18. If \( c \in B_2^{\text{set}}(p) \), then for any \( i \in I \), \( \epsilon_i(c) = 0 \).

Proof. By linearity, it suffices to check that \( \epsilon_i(\partial(g)) = 0 \) for any \( g \in S_3^{\text{set}}(p) \). For simplicity of notation, we assume that \( \text{dom}(g) = \mathcal{P}(s) \) where \( s = \{0, 1, 2, 3\} \). To further simplify, we write "\( g_{i,j} \)" for \( g_{i_{i,j},s} \).

If \( 0 \leq j < k < \ell \leq 3 \), by Lemma 2.17,
\[
\epsilon_i(g \upharpoonright \{j, k, \ell\}) = [g_{j,\ell}^{-1} \circ g_{k,\ell} \circ g_{j,k}]_{G_i}.
\]
Therefore \( \epsilon_i(\partial(g)) \) equals
\[
[ g_{1,3}^{-1} \circ g_{2,3} \circ g_{1,2} ]_{G_i} - [ g_{0,3}^{-1} \circ g_{2,3} \circ g_{0,2} ]_{G_i}
+ [ g_{0,3}^{-1} \circ g_{1,3} \circ g_{0,1} ]_{G_i} - [ g_{0,2}^{-1} \circ g_{1,2} \circ g_{0,1} ]_{G_i}
= [ g_{0,1}^{-1} \circ g_{1,3}^{-1} \circ g_{2,3} \circ g_{1,2} \circ g_{0,1} ]_{G_i} - [ g_{0,3}^{-1} \circ g_{2,3} \circ g_{0,2} ]_{G_i}
+ [ g_{0,3}^{-1} \circ g_{1,3} \circ g_{0,1} ]_{G_i} - [ g_{0,2}^{-1} \circ g_{1,2} \circ g_{0,1} ]_{G_i}
= - [ g_{0,1}^{-1} \circ g_{1,2} \circ g_{0,1} ]_{G_i} - [ g_{0,3}^{-1} \circ g_{2,3} \circ g_{0,2} ]_{G_i} + [ g_{0,1}^{-1} \circ g_{1,3} \circ g_{0,1} ]_{G_i}
+ [ g_{0,1}^{-1} \circ g_{1,3}^{-1} \circ g_{2,3} \circ g_{1,2} \circ g_{0,1} ]_{G_i}
= \left[ (g_{0,1}^{-1} g_{1,2} g_{0,2}) \circ (g_{0,3}^{-1} g_{2,3} g_{0,3}) \circ (g_{0,3}^{-1} g_{1,3} g_{0,1}) \circ (g_{0,1}^{-1} g_{1,3} g_{2,3} g_{1,2} g_{0,1}) \right]_{G_i},
\]
but everything in the last expression cancels out.

\[ \]
By the last lemma, each $\epsilon_i$ induces a well-defined function $\widetilde{\epsilon}_i : H_2(p) \to G_i$.

Now we relate the $\epsilon_i$ maps to the groupoid maps $\chi_{j,i} : G_j \to G_i$. For $i \in I$, let $\psi_i : G_i \to \text{Aut}(SW\xi/\alpha_i, \alpha_i)$ be the map induced by $\psi : \text{Mor}_{G_i}(\alpha_i^1, \alpha_i^1) \to \text{Aut}(SW\xi/\alpha_i, \alpha_i)$, and let $\chi_{j,i} : G_j \to G_i$ be the surjective group homomorphism induced by the function $\chi_{j,i}$ from Lemma 2.14.

Everything coheres:

**Lemma 2.19.** If $i \leq j \in I$ and $f \in S_2(p)$, then $\chi_{j,i}(\epsilon_j(f)) = \epsilon_i(f)$.

**Proof.** For convenience, we assume that $\text{dom}(f) = [2] = \{0, 1, 2\}$. Also, in the proof of this lemma, we write “$f^i_{k,\ell}$” for “$f^i_{\{k,\ell\},[2]}$” (as in Notation 2.15).

**Claim 2.20.** If $i \leq j$, then $\chi_{j,i}(f^j_{k,\ell}) = f^i_{k,\ell}$.

**Proof.** The left-hand side is, by definition, equal to

$$
\chi_{j,i}\left(f^j_{\{k,\ell\},[2]}(\alpha_j(f \upharpoonright \{k, \ell\}))\right) = \left[\pi_{j,i} \left((\alpha_j(f \upharpoonright \{k, \ell\}))\right)\right]_i
$$

(using (6) of Lemma 2.14). But the map $f^j_{\{k,\ell\},[2]}$ is elementary and the functions $\pi_{j,i}$, $(\cdot)_j$, and $[\cdot]_i$ are all definable, so this expression equals

$$
f^j_{\{k,\ell\},[2]}\left((\pi_{j,i}(\alpha_j(f \upharpoonright \{k, \ell\}))\right) = f^i_{\{k,\ell\},[2]}(\alpha_i(f \upharpoonright \{k, \ell\}))
$$

by our choice of the $\alpha_i$ functions such that $\chi_{j,i} \circ \alpha_j = \alpha_i$. But this last expression equals the right-hand side in the Claim. $\dashv$

To prove the lemma, first pick some (any) morphism $g \in \text{Mor}_{G_j}(\alpha_i^1, f_0)$, and note that $\epsilon_j(f)$ is an element of the group $G_j$ which is represented by the following morphism in $\text{Mor}_{G_j}(\alpha_i^1, \alpha_i^1)$:

$$
g^{-1} \circ (f^j_{0,2})^{-1} \circ f^i_{1,2} \circ f^i_{0,1} \circ g.
$$

So $\chi_{j,i}(\epsilon_j(f))$ is represented by the morphism

$$
\chi_{j,i}(g^{-1} \circ (f^j_{0,2})^{-1} \circ f^i_{1,2} \circ f^i_{0,1} \circ g)
$$

which, by the Claim above, equals

$$
\chi_{j,i}(g)^{-1} \circ \chi_{j,i}(f^j_{0,2})^{-1} \circ \chi_{j,i}(f^i_{1,2}) \circ \chi_{j,i}(f^i_{0,1}) \circ \chi_{j,i}(g),
$$

which, by definition, is a representative of $\epsilon_i(f)$. $\dashv$
Let $G$ be the limit of the inverse system of groups $\langle G_i : i \in I \rangle$ with transition maps given by the $\overline{\epsilon_j} : G_j \to G_i$. By Lemma 2.19, the maps $\overline{\epsilon_i}$ induce a group homomorphism $\epsilon : H_2(p) \to G$.

**Lemma 2.21.** The map $\epsilon : H_2(p) \to G$ is injective. In other words, if $c \in Z_2(p)$ and $\epsilon_i(c) = 0$ for every $i \in I$, then $c \in B_2(p)$.

*Proof.* Since $Z_2(p)$ is generated over $B_2(p)$ by all the 2-shells, it is enough to prove this in the case where $c$ is a 2-shell of the form $f_0 - f_1 + f_2 - f_3$, where $f_0$ is a 2-simplex with domain $P([3] \setminus \{a\})$. We will construct a 3-simplex $g : P([3]) \to \mathcal{E}$ such that $\partial g = c$.

Pick some $a_3 \models p((a_0, a_1, a_2))$, so that $(a_0, a_1, a_2, a_3) \models p^{(4)}$. We will construct $g$ so that $g([3]) = \overline{a_{[3]}}$. If $(b, c, d, e)$ is some permutation of $(0, 1, 2, 3)$, then $f_{b,c,d}([b, c]) = f_{b,c,e}([b, c])$ (since $\partial c = 0$), and we can assume that $f_{b,c,d}([b, c]) = \overline{a_{b,c}} = f_{b,c,e}([b, c])$.

As a first step in defining the simplex $g$, for any $\{b, c\} \subseteq \{0, 1, 2, 3\}$, we let $g \upharpoonright \{b, c\} = f \upharpoonright \{b, c, d\}$ (where $d$ is any other element of $[3]$), and we let the maps $g^{(b)}_{[3]} : \overline{a_b} \to \overline{a_{[3]}}$ be the inclusion maps. We take the transition map $g^{(b)}_{[3]}$ (for $b \in [3]$) to be the identity map from $\overline{a_b}$ to itself.

Next we will define the transition maps $g_{[3]}^{bc} : \overline{a_{b,c}} \to \overline{a_{[3]}}$ in such a way as to ensure compatibility with the faces $f_b$. Before doing this, we set some notation. First, we write “$f^{i}_{xy}$” for the set $(f^{i}_{z})_{\{x,y,[3]\} \setminus \{z\}}$ as in Notation 2.15. Similarly, we write

$$\hat{f}_{bc,d} := (f_{d})_{[3] \setminus \{d\}}(\overline{a_{b,c}}).$$

We consider the sets $\overline{a_{b,c}}$ to be 1-simplices in which all of the transition maps are inclusions and the “vertices” are $\overline{a_b}$ and $\overline{a_c}$. This allows us to write “$\alpha_i(\overline{a_{b,c}})$.” For $i \in I$ and $\{b, c\} \subseteq [3]$, let $e_{bc}^i$ be the “edge” $\alpha_i(\overline{a_{b,c}})$.

We define the maps $g_{[3]}^{03}, g_{[3]}^{13},$ and $g_{[3]}^{23}$ to be the identity maps. Then we define the other three edge transition maps $g_{[3]}^{12}, g_{[3]}^{12},$ and $g_{[3]}^{02}$ so that for every $i \in I$,

$$g_{[3]}^{13}(e_{13}^i) g_{[3]}^{23}(e_{23}^i) g_{[3]}^{12}(e_{12}^i) \equiv_{acl(\emptyset)} f_{13,0}^i f_{23,0}^i f_{12,0}^i,$$

$$g_{[3]}^{03}(e_{03}^i) g_{[3]}^{23}(e_{23}^i) g_{[3]}^{02}(e_{02}^i) \equiv_{acl(\emptyset)} f_{03,1}^i f_{23,1}^i f_{02,1}^i,$$

and

$$g_{[3]}^{03}(e_{03}^i) g_{[3]}^{13}(e_{13}^i) g_{[3]}^{01}(e_{01}^i) \equiv_{acl(\emptyset)} f_{03,2}^i f_{13,2}^i f_{01,2}^i.$$
Having specified values according to the three equations above, we let \( g_{[3]}^{i_0}, g_{[3]}^{i_2}, \) and \( g_{[3]}^{i_1} \) be any elementary extensions to the respective domains \( a_{03}, a_{13}, \) and \( a_{23} \).

**Claim 2.22.** For any \( i \in I \),

\[
g_{[3]}^{i_0}(e_{02}^i) g_{[3]}^{i_2}(e_{12}^i) g_{[3]}^{i_1}(e_{01}^i) = f_{02,3}^i f_{12,3}^i f_{01,3}^i.
\]

**Proof.** Note that by stationarity,

\[
g_{[3]}^{i_0}(e_{02}^i) g_{[3]}^{i_2}(e_{12}^i) \equiv f_{02,3}^i f_{12,3}^i,
\]

and to check the Claim, it suffices to show that

\[
\left[ g_{[3]}^{i_0}(e_{02}^i)^{-1} \circ g_{[3]}^{i_2}(e_{12}^i) \circ g_{[3]}^{i_1}(e_{01}^i) \right]_{G_i} = \left[ (f_{02,3}^i)^{-1} \circ f_{12,3}^i \circ f_{01,3}^i \right]_{G_i}.
\]

The right-hand side equals \( \epsilon_i(f_3) \). Since \( \epsilon_i(c) = 0 \),

\[
\epsilon_i(f_3) = \epsilon_i(f_0) - \epsilon_i(f_1) + \epsilon_i(f_2).
\]

Let \( \text{“}g_{bc}\text{”} \) be an abbreviation for \( g_{[3]}^{bc}(e_{bc}^i) \). By applying equations 10, 11, and 12 above (and performing a very similar calculation as in the proof of Lemma 2.18), we get:

\[
\epsilon_i(f_3) = [ g_{13}^{-1} \circ g_{23} \circ g_{12} ]_{G_i} - [ g_{03}^{-1} \circ g_{23} \circ g_{02} ]_{G_i} + [ g_{03}^{-1} \circ g_{13} \circ g_{01} ]_{G_i} = [ g_{01}^{-1} \circ g_{13}^{-1} \circ g_{23} \circ g_{12} \circ g_{01} ]_{G_i} - [ g_{03}^{-1} \circ g_{23} \circ g_{02} ]_{G_i} + [ g_{03}^{-1} \circ g_{13} \circ g_{01} ]_{G_i} = - [ g_{03}^{-1} \circ g_{23} \circ g_{02} ]_{G_i} + [ g_{03}^{-1} \circ g_{13} \circ g_{01} ]_{G_i} + [ g_{01}^{-1} \circ g_{13}^{-1} \circ g_{23} \circ g_{12} \circ g_{01} ]_{G_i} = [ g_{01}^{-1} \circ g_{13}^{-1} \circ g_{23} \circ g_{12} \circ g_{01} ]_{G_i} = [ g_{02}^{-1} \circ g_{12} \circ g_{01} ]_{G_i},
\]

as desired.

Now we must check that this coheres with the types of the given simplices \( f_b \):

**Claim 2.23.** If \( (b, c, d, e) \) is a permutation of \([3]\) with \( 0 \leq b < c < d \leq 3 \), then

\[
g_{[3]}^{bd}(e_{bd}) g_{[3]}^{cd}(e_{cd}) g_{[3]}^{bc}(e_{bc}) \equiv f_{bd,\overline{e}} f_{cd,\overline{e}} f_{bd,\overline{e}}.
\]
Proof. Let 

$$\widetilde{f}_{xy,\hat{e}} := \bigcup_{i \in I} f_{xy,\hat{e}}^i.$$ 

Then Claim 2.23 follows from Claim 2.22 above together with:

**Subclaim 2.24.** If \((x, y, z)\) is any permutation of \((b, c, d)\) with \(x < y\), then \(\text{tp}(f_{xy,\hat{e}}/f_{yz,\hat{e}}f_{xz,\hat{e}})\) is isolated by \(\text{tp}(f_{xy,\hat{e}}/\widetilde{f}_{yz,\hat{e}}\widetilde{f}_{xz,\hat{e}})\).

Proof. Note that \(f_{xy,\hat{e}} \subseteq \text{acl}(f_{yz,\hat{e}}, f_{xz,\hat{e}})\) (in fact, it is in the algebraic closure of the “vertices” \(f_{x,\hat{e}} \subseteq f_{xz,\hat{e}}\) and \(f_{y,\hat{e}} \subseteq f_{yz,\hat{e}}\)). Suppose towards a contradiction that \(h \in f_{xy,\hat{e}}\) but

$$\text{tp}(h/\widetilde{f}_{yz,\hat{e}}\widetilde{f}_{xz,\hat{e}}) \not\subseteq \text{tp}(h/f_{yz,\hat{e}}f_{xz,\hat{e}}).$$

This means that the orbit of \(h\) under \(\text{Aut}(\mathcal{C}/f_{yz,\hat{e}}f_{xz,\hat{e}})\) is smaller than the orbit of \(h\) under \(\text{Aut}(\mathcal{C}/\widetilde{f}_{yz,\hat{e}}\widetilde{f}_{xz,\hat{e}})\). Let \(\hat{h}\) be a name for the orbit of \(h\) under \(\text{Aut}(\mathcal{C}/f_{yz,\hat{e}}f_{xz,\hat{e}})\) as a set. Then

$$\hat{h} \in \text{dcl}(f_{yz,\hat{e}}, f_{xz,\hat{e}}) \setminus \text{dcl}(\widetilde{f}_{yz,\hat{e}}\widetilde{f}_{xz,\hat{e}}).$$

Since \(\hat{h} \in \text{dcl}(f_{yz,\hat{e}}f_{xz,\hat{e}})\), it lies in \(f_{xy,\hat{e}}^i\) for some \(i \in I\) (this is by the maximality condition on our symmetric witnesses \(<W_i : i \in I>\)). Also,

$$f_{xy,\hat{e}}^i \subseteq \text{dcl}(f_{yz,\hat{e}}^i, f_{xz,\hat{e}})$$

due to the fact that \(f_{xy,\hat{e}}^i\) is interdefinable with the set of all morphisms in \(\text{Mor}_G(f_{x,\hat{e}}, f_{y,\hat{e}})\), which can be obtained via composition in \(G_i\) from the corresponding morphisms in \(\text{dcl}(f_{yz,\hat{e}}^i)\) and \(\text{dcl}(f_{xz,\hat{e}})\). But this contradicts the fact that \(\hat{h} \not\in \text{dcl}(\widetilde{f}_{yz,\hat{e}}\widetilde{f}_{xz,\hat{e}})\).

Claim 2.23 implies that for each permutation \((b, c, d, e)\) of \([3]\), we can find an elementary map \(g_{[3],\{3\}}^{b,c,d,e}\) from the “face” \(f_{\hat{e}}([3] \setminus \{e\})\) onto \(\overline{a_{b,c,d}}\) which is coherent with the maps \(g_{[3],\{3\}}^{b,c,d,e}\) and \(g_{[3],\{3\}}^{b,d}\) that we have already defined, and such that \(\partial^i g = f_{\hat{e}}^i\). This completes the proof of Lemma 2.21.

**Lemma 2.25.** The map \(\epsilon : H_2(p) \to G\) is surjective.

Proof. Suppose that \(g\) is any element in \(G\), and that \(g\) is represented by a sequence \(\langle g_i : i \in I \rangle\) such that \(\overline{a_{j,i}}(g_j) = g_i\) whenever \(i \leq j\). We will construct a 2-chain \(c = f - h\) such that \(\epsilon_i(f - h) = g_i\) for every \(i \in I\), which will establish the Lemma. Let \(f : \mathcal{P}(\{2\}) \to \mathcal{C}\) be the 2-simplex such that \(f(s) = \overline{a_{s,e}}^\epsilon\) for every \(s \subseteq \{2\}\) and such that every transition map in \(f\) is an inclusion map. Let \(k_i = \epsilon_i(f)\).
We want to construct \( h : \mathcal{P}(\mathbb{Z}) \rightarrow \mathcal{C} \) such that \( h([2]) = \overline{a_2} \), \( \partial(h) = \partial(f) \), and \( h_{[2]}^s \) is the identity map whenever \( s \subseteq \{0,1\} \) or \( s \subseteq \{1,2\} \).

The only thing left is to specify an elementary map \( h_{[2]}^2 : \overline{a_2} \rightarrow \overline{a_2} \)

fixing \( \overline{a_0} \) and \( \overline{a_2} \) pointwise.

**Claim 2.26.** Suppose that \( h_i \in \text{Mor}_{\mathcal{G}_i}(a_{i_0}^i, a_{i_2}^i) \) is the unique element such that \( [h_i^{-1} \circ h_{12}^i \circ h_{01}^i]_{G_i} = k_i - g_i \). Then

(1) whenever \( i \leq j \), \( \chi_{j,i}(h_j) = h_i \), and

(2) \( \text{tp}(h_0, \ldots, h_i/\overline{a_0}, \overline{a_2}) = \text{tp}(\alpha_0(\overline{a_0}), \ldots, \alpha_i(\overline{a_0})/\overline{a_0}, \overline{a_2}). \)

**Proof.** First we show:

**Subclaim 2.27.** \( \chi_{i+1,i}(h_{12}^{i+1}) = h_{12}^i \) and \( \chi_{i+1,i}(h_{01}^{i+1}) = h_{01}^i \).

**Proof.** We check only the first equation (and the second equation has an identical proof). By (6) of Lemma 2.14,

\[
\chi_{i+1,i}(h_{12}^{i+1}) = [\pi_{i+1,i}(\langle h_{12}^{i+1} \rangle_{i+1})]_i = [\pi_{i+1,i}(\langle h_{12}^i(\alpha_{i+1}(\overline{a_1})) \rangle_{i+1})]_i
\]

\[
= h_{12}^i ([\pi_{i+1,i}(\langle \alpha_{i+1}(\overline{a_1}) \rangle_{i+1})]_i) = h_{12}^i (\chi_{i+1,i}(\alpha_{i+1}(\overline{a_1})))
\]

\[
= h_{12}^i (\alpha_i(\overline{a_1})) = h_{12}^i.
\]

Note that it is enough to prove (1) of the Claim for every pair \((i,j)\) where \( j = i + 1 \). We apply \( \chi_{i+1,i} \) to both sides of the equation \( \chi_{i+1,i} \) applied to both sides of the equation

\[
[h_{12}^{-1} \circ h_{12}^i \circ h_{01}^i]_{G_{i+1}} = k_{i+1} - g_{i+1}.
\]

On the right-hand side, this yields

\[
(14) \quad \chi_{i+1,i}(k_{i+1} - g_{i+1}) = k_i - g_i.
\]

On the left-hand side, using the Subclaim, we get

\[
(15) \quad [\chi_{i+1,i} \left( h_{12}^{-1} \circ h_{12}^i \circ h_{01}^i \right)]_{G_i} = [\chi_{i+1,i}(h_{i+1})^{-1} \circ h_{12}^i \circ h_{01}^i]_{G_i}.
\]

So putting together Equations 14 and 15, we get that

\[
[\chi_{i+1,i}(h_{i+1})^{-1} \circ h_{12}^i \circ h_{01}^i]_{G_i} = k_i - g_i.
\]

But \( h_i \) is the unique morphism in \( \mathcal{G}_i \) such that \( [h_i^{-1} \circ h_{12}^i \circ h_{01}^i]_{G_i} = k_i - g_i \), so part (1) of the Claim follows.

We prove part (2) by induction on \( i \in I \). The base case follows from

\[
\text{tp}(\langle h_0 \rangle_0/\overline{a_0}, \overline{a_2}) = \text{tp}(\langle \alpha_0(\overline{a_0}) \rangle_0/\overline{a_0}, \overline{a_2}).
\]
(which is true simply because both elements belong to \( SW(a_1^i, a_2^i) \)). If (2) is true for \( i \), then to prove it for \( i + 1 \), it is enough to check that

\[
\text{tp}(\langle h_{i+1} \rangle_{i+1}/\overline{a_0}, \overline{a_2}) = \text{tp}(\langle \alpha_{i+1}(\overline{a_0}) \rangle_{i+1}/\overline{a_0}, \overline{a_2}),
\]

since all the other elements are in the definable closure of \( h_i \) and \( \alpha_i(\overline{a_0}) \) via the maps \( \chi_{k,\ell} \). To see this, first note that

\[
\text{tp}(\langle h_{i+1} \rangle_{i+1}/\overline{a_0}, \overline{a_2}) = \text{tp}(\langle \alpha_{i+1}(\overline{a_0}) \rangle_{i+1}/\overline{a_0}, \overline{a_2})
\]

just because both elements belong to \( SW(a_1^{i+1}, a_2^{i+1}) \). By part (1) of the Claim, \( \pi_{i+1, i}(\langle h_{i+1} \rangle_{i+1}) = \langle h_i \rangle_{i} \) and by the way we chose the \( \alpha \) functions, \( \pi_{i+1, i}(\langle \alpha_{i+1}(\overline{a_0}) \rangle_{i+1}) = \langle \alpha_i(\overline{a_0}) \rangle_{i} \). Since the function \( \pi_{i+1, i} \) is definable, Equation 16 follows.

Given the elements \( h_i \) as in the Claim above, we let \( h_0^{i_2} : \overline{a_0} \to \overline{a_0} \) be any elementary map that fixes \( \overline{a_0} \cup \overline{a_2} \) pointwise and maps each element \( \alpha_i(\overline{a_0}) \) to \( h_i \). Then \( \epsilon_i(f - h) = k_i - (k_i - g_i) = g_i \), as desired.

By Lemmas 2.21 and 2.25, \( H_2(p) \cong G \). To finish the proof of Theorem 2.1, we just need to show:

**Lemma 2.28.** \( G \cong \text{Aut}(\overline{a_0a_1}/\overline{a_0}, \overline{a_1}) \)

**Proof.** Note that \( \text{Aut}(\overline{a_0a_1}/\overline{a_0}, \overline{a_1}) \) is the limit of the groups \( \text{Aut}(SW_i/\overline{a_0}, \overline{a_1}) \) via the transition maps \( \rho_{j,i} : \text{Aut}(SW_j/\overline{a_0}, \overline{a_1}) \to \text{Aut}(SW_i/\overline{a_0}, \overline{a_1}) \), due to the maximality condition that every element of \( a_0a_1 \) lies in one of the symmetric witnesses \( SW_i \). Also, part (8) of Lemma 2.14 implies that we have a commuting system

\[
\begin{array}{ccc}
G_j & \xrightarrow{\overline{\chi}_{j,i}} & G_i \\
\downarrow{\overline{\psi}_j} & & \downarrow{\overline{\psi}_i} \\
\text{Aut}(SW_j/\overline{a_0}, \overline{a_1}) & \xrightarrow{\rho_{j,i}} & \text{Aut}(SW_i/\overline{a_0}, \overline{a_1})
\end{array}
\]

But the maps \( \overline{\psi}_i \) are all isomorphisms, so taking limits we get an isomorphism from \( G \) to \( \text{Aut}(\overline{a_0a_1}/\overline{a_0}, \overline{a_1}) \).

3. Any profinite abelian group can occur as \( H_2(p) \)

In this section, we construct a family of examples which prove the following:

**Theorem 3.1.** For any profinite abelian group \( G \), there is a type \( p \) in a stable theory \( T \) such that \( H_2(p) \cong G \). In fact, we can build the theory \( T \) to be totally categorical.
Together with Theorem 2.1 from the previous section, this shows that the groups that can occur as $H_2(p)$ for a type $p$ in a stable theory are precisely the profinite abelian groups.

For the remainder of this section, we fix a profinite abelian group $G$ which is the inverse limit of the system $\langle H_i : i \in I \rangle$, where each $H_i$ is finite and abelian, $(I, \leq)$ is a directed set, and $G$ is the limit along the surjective group homomorphisms $\varphi_{j,i} : H_j \rightarrow H_i$ (for every pair $i \leq j$ in $I$). The language $L$ of $T$ will be as follows: there will be a sort $G_i$ for each $i \in I$, and function symbols $\chi_{j,i} : G_j \rightarrow G_i$ for every pair $i \leq j$. The theory $T$ will say, in the usual language of categories, that each $G_i$ is a connected groupoid with infinitely many objects, and there will be separate composition symbols for each sort $G_i$. Also, $T$ says that $G_i$ is a groupoid such that each vertex group $\text{Mor}_{G_i}(a_i, a_i)$ is isomorphic to the group $H_i$. For convenience, pick some arbitrary $a_i \in \text{Ob}(G_i)$ and some group isomorphism $\xi_i : G_i \rightarrow \text{Mor}_{G_i}(a_i, a_i)$ (but the $\xi_i$'s are not a part of any model of $T$). Then the last requirement we make on $T$ is that the function symbols $\chi_{j,i}$ define full functors from $G_j$ onto $G_i$ which induce bijections between the corresponding collections of objects, and such that for every pair $i \leq j$, the following diagram commutes:

$$
\begin{array}{ccc}
H_j & \xrightarrow{\varphi_{j,i}} & H_i \\
\downarrow{\xi_i} & & \downarrow{\xi_i} \\
\text{Mor}_{G_j}(a_j, a_j) & \xrightarrow{\chi_{j,i}} & \text{Mor}_{G_i}(a_i, a_i)
\end{array}
$$

(In other words, the functors $\chi_{j,i}$ are just “isomorphic copies” the group homomorphisms $\varphi_{j,i}$.)

**Lemma 3.2.** The theory $T$ described above is complete and admits elimination of quantifiers. If we further assume that the language is multi-sorted and that every element of a model must belong to one of the sorts $G_i$, then $T$ is totally categorical.

*Proof.* If the language is multi-sorted, then since the groupoids $G_i$ are all connected and there are bijections between the object sets of the various $G_i$, the isomorphism class of a model of $T$ is determined by the cardinality of the object set of some (any) $G_i$. This shows that $T$ is totally categorical, hence $T$ is complete.

For quantifier elimination, it suffices to show the following: for any two models $M_1$ and $M_2$ of $T$ with a common substructure $A$ and any sentence $\sigma$ with parameters from $A$ of the form $\sigma = \exists x \varphi(x; \overline{a})$ where $\varphi$ is quantifier-free, if $M_1 \models \sigma$, then $M_2 \models \sigma$. (See Theorem 8.5 of [13].) In this situation, let $\text{cl}(A)$ denote the submodel of $M_1$ (and of
generated by \( A \), and in case \( A = \emptyset \), let \( \text{cl}(A) = \emptyset \). Then if \( M_1 \models \sigma \) as above, at least one of the following is true:

1. \( \varphi(x; \bar{a}) \) is satisfied by some \( x \) in \( \text{cl}(A) \);
2. \( \varphi(x; \bar{a}) \) is satisfied by some morphism between two objects in \( \text{cl}(A) \);
3. For some \( i \in I \), \( \varphi(x; \bar{a}) \) is satisfied by any object in \( \mathcal{G}_i \) outside of \( \text{cl}(A) \);
4. For some \( i \in I \), \( \varphi(x; \bar{a}) \) is satisfied by any morphism in \( \mathcal{G}_i \) which goes from [or to] some particular \( b \in \text{cl}(A) \) and goes to [or from] any object in \( \mathcal{G}_i \) outside of \( \text{cl}(A) \); or else
5. For some \( i \in I \), \( \varphi(x; \bar{a}) \) is satisfied by any morphism in \( \mathcal{G}_i \) whose source and target are both outside of \( \text{cl}(A) \).

In each of the five cases above, it is straightforward to check that there is an \( x \) realizing \( \varphi(x; \bar{a}) \) in \( M_2 \) as well (for the last three cases we use the fact that \( \text{Ob}(\mathcal{G}_i) \) is infinite).

\[ \square \]

**Remark 3.3.** If \( A \subseteq \mathcal{G}_i \), then we say that \( b \in \text{Ob}(\mathcal{G}_i) \) is connected to \( A \) if either \( b \in A \) or \( b \) is the source or target of a morphism in \( A \). By elimination of quantifiers, it follows that for any \( A \subseteq \mathcal{G}_i \), \( \text{acl}(A) \cap \mathcal{G}_i \) is the union of all objects \( b \) that are connected to \( A \) plus all morphisms \( f \in \text{Mor}_{\mathcal{G}_i}(b,c) \) such that \( b \) and \( c \) are connected to \( A \).

Because of the functors \( \chi_{j,i} \), it follows that for any \( a \) in any \( \mathcal{G}_i \), \( \text{acl}(a) \) actually contains objects and morphisms from each of the groupoids \( \mathcal{G}_j \). But for any \( A \subseteq \mathcal{C} \), we can write \( \text{acl}(A) \) in the “standard form” \( \text{acl}(A) = \text{acl}(A_0) \) for some \( A_0 \subseteq \text{Ob}(\mathcal{G}_0) \), and:

1. \( \text{acl}(A_0) \cap \text{Ob}(\mathcal{G}_i) = \chi_{i,0}^{-1}(A_0) \), and
2. \( \text{acl}(A_0) \cap \text{Mor}(\mathcal{G}_i) \) is the collection of all \( f \in \text{Mor}_{\mathcal{G}_i}(b,c) \) where \( b, c \in \text{acl}(A_0) \).

**Lemma 3.4.** *The theory \( T \) has weak elimination of imaginaries in the sense of [12]: for every formula \( \varphi(\bar{x}, \bar{a}) \) defined over a model \( M \) of \( T \), there is a smallest algebraically closed set \( A \subseteq M \) such that \( \varphi(\bar{x}, \bar{a}) \) is equivalent to a formula with parameters in \( A \).*

**Proof.** By Lemma 16.17 of [12], it suffices to prove the following two statements:

1. There is no strictly decreasing sequence \( A_0 \supseteq A_1 \supseteq \ldots \), where every \( A_i \) is the algebraic closure of a finite set of parameters; and
2. If \( A \) and \( B \) are algebraic closures of finite sets of parameters in the monster model \( \mathcal{C} \), then \( \text{Aut}(\mathcal{C}/A \cap B) \) is generated by \( \text{Aut}(\mathcal{C}/A) \) and \( \text{Aut}(\mathcal{C}/B) \).
Statement 1 follows immediately from the characterization of algebraically closed sets in Remark 3.3 above (that is, algebraic closures of finite sets are equivalent to algebraic closures of finite subsets of \( \text{Ob}(\mathcal{G}_0) \)).

To check statement 2, suppose that \( \sigma \in \text{Aut}(\mathcal{C}/A \cap B) \), and assume that \( A = \text{acl}(A_0) \) and \( B = \text{acl}(B_0) \) where \( A_0, B_0 \subseteq \text{Ob}(\mathcal{G}_0) \). Note that any permutation of \( \text{Ob}(\mathcal{G}_0)/(\mathcal{C}/A_0 \cap B_0) \) which fixes \( A_0 \) can be extended to an automorphism of \( \text{Aut}(\mathcal{C}/A) \), and likewise for \( B_0 \) and \( B \). So as a first step, we can use the fact that Sym(\( \text{Ob}(\mathcal{G}_0)/(\mathcal{C}/A_0 \cap B_0) \)) is generated by Sym(\( \text{Ob}(\mathcal{G}_0)/(\mathcal{C}/A_0) \)) and Sym(\( \text{Ob}(\mathcal{G}_0)/(\mathcal{C}/B_0) \)) to find an automorphism \( \tau \in \text{Aut}(\mathcal{C}/A) \) such that \( \tau \) is in the subgroup generated by \( \text{Aut}(\mathcal{C}/A_0) \) and \( \text{Aut}(\mathcal{C}/B) \) and \( \sigma \circ \tau^{-1} \) fixes \( \text{Ob}(\mathcal{G}_0) \) (and hence \( \text{Ob}(\mathcal{G}_i) \) for every \( i \)) pointwise.

Finally, we need to deal with the morphisms. We claim that there is a map \( \sigma_0^A \in \text{Aut}(\mathcal{C}/A) \) which fixes \( \text{Ob}(\mathcal{G}_0) \) pointwise and such that for any \( f \in \text{Mor}_{\mathcal{G}_i}(b,c) \) such that at least one of \( b \) and \( c \) do not lie in \( A \), \((\sigma_0^A \circ \tau)(f) = \sigma(f) \). (The idea is to use the recipe for constructing object-fixing automorphisms described in subsection 4.2 of [4], using a basepoint \( a_0 \in A \).) In fact, by the same argument we can also assume that for every \( i \in I \) and for any \( f \in \text{Mor}_{\mathcal{G}_i}(b,c) \) such that at least one of \( b \) and \( c \) do not lie in \( A \), \((\sigma_0^A \circ \tau)(f) = \sigma(f) \). Similarly, there is a map \( \sigma_0^B \in \text{Aut}(\mathcal{C}/B) \) which fixes \( \text{Ob}(\mathcal{G}_0) \) pointwise and for any \( i \in I \), \( \sigma_0^B \) only moves morphisms in \( \text{Mor}_{\mathcal{G}_i}(b,c) \) where \( b \) and \( c \) are both in \( A \setminus (A \cap B) \), and such that \( \sigma_0^B \circ \sigma_0^A \circ \tau = \sigma \).

\[\square\]

**Lemma 3.5.** If \( a^0, a^1 \in \text{Ob}(\mathcal{G}_i) \), then

\[
\text{acl}^e(a^0, a^1) = \text{dcl}^e \left( \bigcup_{i,j \in I, i \leq j} \text{Mor}_{\mathcal{G}_j}(a^0_j, a^1_j) \right),
\]

where \( a^\ell_j = \chi_{j,i}^{-1}(a^\ell) \).

**Proof.** Suppose \( g \in \text{acl}^e(a^0, a^1) \). Then \( g = b/E \) for some \( (a^0, a^1) \)-definable finite equivalence relation \( E \). By Lemma 3.4, there is a finite tuple \( \bar{d} \in \mathcal{C} \) (in the home sort) such that \( b/E \) is definable over \( \bar{d} \) and \( \bar{d} \) has a minimal algebraic closure. If the set \( \text{acl}(\bar{d}) \) contained an object \( a \) of \( \mathcal{G}_0 \) other than \( \chi_{i,0}(a^0) \) and \( \chi_{j,0}(a^1) \), then (by quantifier elimination) \( \text{acl}(\bar{d}) \) would have an infinite orbit under \( \text{Aut}(\mathcal{C}/a^0, a^1) \), and so \( E \) would have infinitely many classes, a contradiction. So by Remark 3.3, the set \( \bar{d} \), and hence \( b/E \) is definable over the union of the morphism sets \( \text{Mor}_{\mathcal{G}_j}(a^0_j, a^1_j) \).

\[\square\]
From now on, we assume that all algebraic and definable closures are computed in $T^e$, not just in the home sort.

**Lemma 3.6.** If $a^0, a^1 \in \text{Ob}(\mathcal{G}_i)$, then for any two $f, g \in \text{Mor}_{\mathcal{G}_i}(a^0, a^1)$,
\[
\text{tp}(f/\text{acl}(a^0), \text{acl}(a^1)) = \text{tp}(g/\text{acl}(a^0), \text{acl}(a^1)).
\]

**Proof.** Using the same procedure as described in subsection 4.2 of [4], we can construct an automorphism $\sigma$ of $\mathcal{C}$ fixing $\text{Ob}(\mathcal{G}_i)$, $\text{Mor}_{\mathcal{G}_i}(a^0, a^0)$, and $\text{Mor}_{\mathcal{G}_i}(a^1, a^1)$ pointwise while mapping $f$ to $g$. (In the construction of [4], the “basepoint” $a_0$ there can be chosen to be $a^0$ here, and then condition (5) of the construction plus the fact that $\mathcal{G}_i$ is abelian implies that $\text{Mor}_{\mathcal{G}_i}(a^1, a^1)$ is fixed.) In fact, it is easy to see that we can even ensure that $\sigma$ fixes $\text{Mor}_{\mathcal{G}_i}(\chi_{j,i}^{-1}(a^0), \chi_{j,i}^{-1}(a^0))$ and $\text{Mor}_{\mathcal{G}_i}(\chi_{j,i}^{-1}(a^1), \chi_{j,i}^{-1}(a^1))$ pointwise, so by Lemma 3.5, $\sigma$ fixes $\text{acl}(a^0) \cup \text{acl}(a^1)$ pointwise. $\dashv$

Let $p = \text{stp}(a_0)$ for some (any) $a_0 \in \text{Ob}(\mathcal{G}_0)$.

**Proposition 3.7.** $H_2(p) \cong G$.

**Proof.** Pick $(a^0, a^1, a^2) \models p^{(3)}$. By Theorem 2.1 (the “Hurewicz theorem”), it is enough to show that $\text{Aut}(\overline{a^0a^1}/\overline{a^0}, \overline{a^1}) \cong G$. For ease of notation, let $a^k_i = \chi_{i,0}^{-1}(a^k)$ for $k = 0, 1, 2$. By Lemma 3.5 and the fact that any morphism in $\text{Mor}_{\mathcal{G}_i}(a^0_i, a^1_i)$ is a composition of morphisms in $\text{Mor}_{\mathcal{G}_i}(a^0_i, a^2_i)$ and $\text{Mor}_{\mathcal{G}_i}(a^2_i, a^1_i)$, it follows that the set $\overline{a^0a^1}$ is interdefinable with $\bigcup_{i \in I} \text{Mor}_{\mathcal{G}_i}(a^0_i, a^1_i)$.

So $\text{Aut}(\overline{a^0a^1}/\overline{a^0}, \overline{a^1})$ is the inverse limit of the groups $\text{Aut}(\text{Mor}_{\mathcal{G}_i}(a^0_i, a^1_i)/\overline{a^0}, \overline{a^1})$ under the natural homomorphisms
\[
\rho_{j,i} : \text{Aut}(\text{Mor}_{\mathcal{G}_j}(a^0_j, a^1_j)/\overline{a^0}, \overline{a^1}) \to \text{Aut}(\text{Mor}_{\mathcal{G}_i}(a^0_i, a^1_i)/\overline{a^0}, \overline{a^1})
\]
induced by the fact that $\text{Mor}_{\mathcal{G}_i}(a^0_i, a^1_i)$ is in the definable closure of $\text{Mor}_{\mathcal{G}_j}(a^0_j, a^1_j)$ when $j \geq i$.

By the way we defined our theory $T$, we can select a system of group isomorphisms $\lambda_i : H_i \to \text{Mor}_{\mathcal{G}_i}(a^0_i, a^1_i)$ for $i \in I$ such that the following diagram commutes:

\[
\begin{array}{ccc}
H_j & \xrightarrow{\varphi_{j,i}} & H_i \\
\downarrow{\lambda_j} & & \downarrow{\lambda_i} \\
\text{Mor}_{\mathcal{G}_j}(a_j, a_j) & \xrightarrow{\chi_{j,i}} & \text{Mor}_{\mathcal{G}_i}(a_i, a_i)
\end{array}
\]

To finish the proof of the Proposition, it is enough to find a system of group isomorphisms
\[
\sigma_i : H_i \to \text{Aut}(\text{Mor}_{\mathcal{G}_i}(a^0_i, a^1_i)/\overline{a^0}, \overline{a^1})
\]
such that the following diagram commutes:

\[
\begin{array}{ccc}
H_j & \xrightarrow{\varphi_{j,i}} & H_i \\
\downarrow{\sigma_j} & & \downarrow{\sigma_i} \\
\text{Aut}(\text{Mor}_{\mathcal{G}_j}(a_j^0, a_j^1)/a_0^0, a_1^1) & \xrightarrow{\rho_{j,i}} & \text{Aut}(\text{Mor}_{\mathcal{G}_i}(a_i^0, a_i^1)/a_0^0, a_1^1)
\end{array}
\]

(Then by the discussion above, \(\text{Aut}(a_0^0a_1^1/a_0^0, a_1^1)\) will be isomorphic to the inverse limit of the groups \(H_i\), which is \(G\).)

We define the maps \(\sigma_i\) so that for any \(h \in H_i\) and any \(g \in \text{Mor}_{\mathcal{G}_i}(a_i^0, a_i^1)\),

\[\left[\sigma_i(h)\right](g) = \lambda_i(h) \circ g.\]

(Note that this rule determines a unique elementary permutation \(\sigma\) of \(\text{Mor}_{\mathcal{G}_i}(a_i^0, a_i^1)\) fixing \(\text{acl}(a_0^0) \cup \text{acl}(a_1^1)\) pointwise.) This is a group homomorphism since

\[\left[\sigma_i(h_1h_2)\right](g) = \lambda_i(h_1h_2) \circ g = \lambda_i(h_1) \circ \lambda_i(h_2) \circ g = \left[\sigma_i(h_1) \circ \sigma_i(h_2)\right](g).\]

Clearly \(\sigma_i\) is injective, and it is surjective because of the following:

**Claim 3.8.** For any \(f\) and \(g\) in \(\text{Mor}_{\mathcal{G}_i}(a_i^0, a_i^1)\), there is a unique elementary permutation \(\sigma\) of \(\text{Mor}_{\mathcal{G}_i}(a_i^0, a_i^1)\) sending \(f\) to \(g\) and fixing \(\text{acl}(a_0^0) \cup \text{acl}(a_1^1)\) pointwise.

**Proof.** If \(f = h \circ g\) for \(h \in \text{Mor}_{\mathcal{G}_i}(a_i^1, a_i^1)\), then \(\sigma(f')\) must equal \(h \circ f'\) for any \(f' \in \text{Mor}_{\mathcal{G}_i}(a_i^0, a_i^1)\). \(\dashv\)

Finally, we must check that the maps \(\sigma_i\) commute with \(\varphi_{j,i}\) and \(\rho_{j,i}\). Pick any \(j \geq i, h \in H_j\) and \(f \in \text{Mor}_{\mathcal{G}_j}(a_j^0, a_j^1)\). On the one hand,

\[\rho_{j,i}(\sigma_j(h))(f) = \chi_{j,i}(\sigma_j(h)(f'))\text{, where }\chi_{j,i}(f') = f\]

\[= \chi_{j,i}(\lambda_j(h) \circ f') = \chi_{j,i}(\lambda_j(h)) \circ \chi_{j,i}(f') = \chi_{j,i}(\lambda_j(h)) \circ f.\]

On the other hand,

\[\left[\sigma_i(\varphi_{j,i}(h))\right](f) = \lambda_i(\varphi_{j,i}(h)) \circ f = \chi_{j,i}(\lambda_j(h)) \circ f.\]

These last two equations show that \(\rho_{j,i} \circ \sigma_j = \sigma_i \circ \varphi_{j,i}\), as desired. \(\dashv\)

**Remark 3.9.** These examples also show that homology groups of types are not always preserved by nonforking extensions. In the example above, if \(A\) is some algebraically closed parameter set containing a point in \(p(C)\) and \(q\) is the nonforking extension of \(p\) over \(A\), then \(q\) has 4-amalgamation, and so by Fact 1.8 above, \(H_2(q) = 0\).
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