Convergence in Topological Spaces.
Nets.

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Abstract
Sequences are not always sufficient to describe convergence in topological spaces. In what follows we shall introduce a generalisation to sequence which can satisfactorily describe convergence in any topological space.

1 Preliminaries.
It is well-known that many topological properties of metric spaces can be described using sequences (see [Ru, Chapters 3-4]). For instance:

Theorem 1.1.
1. A subset $A$ of a metric space $X$ is open if and only if all sequences $\langle x_n \rangle$ in $X$ which converge to a point $x \in A$ are eventually in $A$.

2. A subset $A$ of a metric space $X$ is closed if and only if, for all sequences $\langle x_n \rangle$ in $A$, if $\lim_{n \to \infty} x_n = x$, then $x \in A$.

3. If $A$ is a subset of a metric space, then $x \in \overline{A}$ if and only if there exists a sequence $\langle x_n \rangle$ in $A$ which converges to $x$.

4. A function $f : X \to Y$ between metric spaces is continuous if and only if, whenever $\langle x_n \rangle$ is a sequence which converges to $x$ in $X$, then $\langle f(x_n) \rangle$ converges to $f(x)$ in $Y$.

As we shall see, in general topological spaces the above statements are no longer true unless certain conditions are satisfied. We start by extending the definition of convergence (of sequences) to general topological spaces.

Definition 1.2. A sequence in a topological space $X$ is a function $f : \mathbb{N} \to X$. It is common practice to denote a sequence by $\langle x_n \rangle$ meaning that, for $n \in \mathbb{N}$, $f(n) = x_n$.

Definition 1.3. A sequence $\langle x_n \rangle$ in a topological space $X$ converges to $L \in X$ if, for all (basic) neighbourhood $U$ of $L$, there exists $n_U \in \mathbb{N}$ such that, whenever $m > n_U$, we have $x_m \in U$.

Remembering proof of Theorem 1.1 it should be clear that when we can find countably many “arbitrarily small” neighbourhoods around all points of $X$, we can still replicate this proof in a topological space. In order to do so we start with a definition which formalises the concept or “arbitrarily small” neighbourhood to topological spaces.

Definition 1.4. A topological space $X$ is first countable if every point has a countable neighbourhood base.
For first countable spaces the four statements of Theorem 1.1 are true. The proofs are very similar to those given for metric spaces. Here we prove only one of the propositions (the other three are left as exercises) in order to illustrate which kind of changes are needed.

**Theorem 1.5.** Let $X$ be a first countable space and $A \subseteq X$. Then, $x \in \overline{A}$ if and only if there exists a sequence $\langle x_n \rangle$ in $A$ converging to $x$.

**Proof.** Let $x \in \overline{A}$ and let $\{U_n : n \in \mathbb{N}\}$ be a countable base of neighbourhoods of $x$. Let $V_i := \bigcap_{k=1}^{i} U_k$. It is not difficult to see that $\{V_n : n \in \mathbb{N}\}$ is also a countable base of neighbourhoods of $x$ and that:

$$U_1 \supset U_2 \supset U_3 \supset \ldots.$$ 

Since $x \in \overline{A}$, for all $i \in \omega$, $V_i \cap A \neq \emptyset$. So, for all $i \in \mathbb{N}$, we can pick $x_i \in V_i \cap A$. The sequence $\langle x_n \rangle$ obtained this way clearly converges to $x$.

Conversely, if $\langle x_n \rangle$ converges to $x$, then, every neighbourhood $U$ of $x$ contains infinitely many points of this sequence. Hence $U$ it has not empty intersection with $A$, i.e. $x \in \overline{A}$. ☐

The previous proof relies on the fact that all points of $X$ have a countable neighbourhood base. If this is not the case it may happen\(^1\) that sequences are no longer sufficient to describe the topology of the space. The following examples (which is left as an exercise) illustrate this.

**Example 1.6.** Let $X = \mathbb{R}^\mathbb{R}$ with the product topology. Let us define a subset $E$ of $\mathbb{R}^\mathbb{R}$ as follows:

$$E = \{f \in \mathbb{R}^\mathbb{R} : f(x) = 0 \text{ or } f(x) = 1 \text{ and } f(x) = 0 \text{ only for finitely many } x \in \mathbb{R}\}.$$ 

Let $g$ be the function in $\mathbb{R}^\mathbb{R}$ which is identically 0 and $U(g)$ a basic neighbourhood of $g$,

$$U(g) = \{h \in \mathbb{R}^\mathbb{R} : |h(y) - g(y)| < \varepsilon \text{ if } y \in F\},$$

where $F$ is some finite subset of $\mathbb{R}$.

The set $U(g) \cap E$ is not empty, since it contains the function which is 0 on $E$ and 1 elsewhere, then $g \in E$.

However, no sequence in $E$ converges to $g$.

# 2 Nets

In this section we shall introduce the concept of net and show that this is a valid substitute to sequences to describe convergence in topological spaces.

**Definition 2.1.** Let $I$ be a set partially ordered by $\leq$. We say that $I$ is *directed* by $\leq$ if, for every, $\alpha, \beta \in I$, there exists $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

**Definition 2.2.** A net in a topological space $X$ is a function $f$ from a directed set $I$ to $X$. Usually, a net is denoted by $\langle x_\alpha \rangle_{\alpha \in I}$ meaning that $f(\alpha) = x_\alpha$.

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\(^{1}\)One may think that being first countable is a necessary and sufficient condition in order for sequences to be enough to describe the topology. However, this is not true. Many mathematicians have studied the problem of characterising the topological spaces for which sequences suffices in order to describe their topology and obtained a somehow wider class than that of first countable spaces.
Definition 2.3. A net \( \langle x_\alpha \rangle_{\alpha \in I} \) in a topological space \( X \), is said to be frequently in a subset \( A \) of \( X \) if, for all \( \alpha_0 \in I \), there exists some \( \alpha \geq \alpha_0 \) such that \( x_\alpha \in A \). The same net is said to be eventually in \( A \) if there exists some \( \alpha_0 \in I \) such that, for all \( \alpha \geq \alpha_0 \), \( x_\alpha \in A \).

Definition 2.4. A net \( \langle x_\alpha \rangle_{\alpha \in I} \), in a topological space \( X \) is said to have a point \( x \) of \( X \) as a cluster point if the net is frequently in every neighbourhood of \( x \). If the net is eventually in every neighbourhood of \( x \), then it is said to converge to \( x \) and \( x \) is said to be a limit of the net. In this case we write \( x = \lim_{\alpha \in I} x_\alpha \). Sometimes, if there is no danger of confusion we simply write \( x = \lim_{\alpha} x_\alpha \) or also \( x_\alpha \rightarrow x \). When the limit is unique we write \( x = \lim x_\alpha \) instead of \( \{ x \} = \lim x_\alpha \).

Nets are a useful tool to describe convergence in metric spaces, however to formulate the concept of subnet is rather cumbersome.

Definition 2.5. Let \( I \) and \( J \) be directed sets. A function \( f : J \rightarrow I \) is said to be increasing and cofinal if:

1. For all \( \alpha, \beta \in J \) such that \( \alpha \leq \beta \), \( f(\alpha) \leq f(\beta) \), and;
2. For all \( \delta \in I \), there exists \( \alpha \in J \) such that \( f(\alpha) \geq \delta \).

Definition 2.6. Let \( \langle x_\alpha \rangle \) be a net in \( X \) indexed by \( I \) and let \( f : I \rightarrow X \) be the function such that \( f(\alpha) = x_\alpha \). If \( g : J \rightarrow I \) is an increasing cofinal function between the directed sets \( J \) and \( I \), then the composition \( f \circ g : J \rightarrow X \) is called a subnet of \( \langle x_\alpha \rangle \).

Theorem 2.7. Let \( \langle x_\alpha \rangle \) be a net in a topological space \( X \), and \( \langle x_{\alpha_0} \rangle \) a subnet of \( \langle x_\alpha \rangle \) (by this notation we mean that the increasing cofinal function which defines the subnet maps \( \alpha \) into \( \alpha_0 \)). Then the following statements hold:

1. If \( x \) is a cluster point of \( \langle x_{\alpha_0} \rangle \), then \( x \) is also a cluster point of \( \langle x_\alpha \rangle \);
2. If \( x \) is a limit of \( \langle x_\alpha \rangle \) then \( x \) is a limit of \( \langle x_{\alpha_0} \rangle \);
3. If \( x \) is a cluster point of \( \langle x_\alpha \rangle \) then there exists a subnet \( \langle x_{\alpha_0} \rangle \) of \( \langle x_\alpha \rangle \), which admits \( x \) as a limit.

Proof. 1. Let \( U \) be a neighbourhood of \( x \) and \( \alpha_0 \in I \). Since \( \langle x_{\alpha_0} \rangle \) is a subnet of \( \langle x_\alpha \rangle \) there is \( \beta_0 \in J \) such that \( \alpha_0 \leq \beta_0 \). Since \( x \) is a cluster point of \( \langle x_{\alpha_0} \rangle \), there exists \( \beta' \in J \) such that \( \alpha_0 \leq \beta' \) and \( x_{\alpha_0 \beta'} \in U \). This means that \( x \) is a cluster point of \( \langle x_\alpha \rangle \).
2. If \( x \) is a limit of \( \langle x_\alpha \rangle \), then, for all neighbourhood \( U \) of \( x \), \( \langle x_\alpha \rangle \) is eventually in \( U \). In particular, \( \langle x_{\alpha_0} \rangle \) is eventually in \( U \).
3. Let \( x' \) be a cluster point of \( \langle x_\alpha \rangle \). Consider

\[ K = \{ (\beta, U) : \beta \in I \text{ and } U \text{ is a neighbourhood of } x, \ x_\beta \in U \}. \]

We define a partial order on \( K \) as follows:

\[ (\beta_1, U_1) \leq (\beta_2, U_2) \text{ if } \beta_1 \leq \beta_2 \text{ and } U_1 \supseteq U_2. \]

It is not difficult to check that \( K \) is directed by \( \leq \). We define an increasing cofinal map between \( K \) and \( I \) by \( (\beta, U) \mapsto \beta \) and a subnet of \( \langle x_\alpha \rangle \) by \( x_{\alpha_{\beta, U}} = x_\alpha \). For all neighbourhood \( U \) of \( x \), there exists some \( \beta_0 \) in \( I \) such that \( x_{\beta_0} \in U \), so for all \( (\beta_0, U_0) \leq (\beta, U) \), \( x_{\beta_0} \in U \subseteq U_0 \), i.e. \( x \in \lim x_{\alpha_{\beta, U}} \), as requested.

\[ \square \]
Theorem 2.8. Let $A$ be a subset of a topological space $X$. A point $x$ of $X$ belongs to $\overline{A}$, if and only if there exists a net in $A$ converging to $x$.

Proof. Let $S = \langle x_\alpha \rangle_{\alpha \in I}$ be a net in $A$ converging to $x$. Then any neighbourhood of $x$ contains points of $A$, so $x$ is in $\overline{A}$. Conversely, let $x \in \overline{A}$ and let $N_x$ be the set of the neighbourhoods of $x$. The set $N_x$ is directed by the order $\leq$ given by:

$$U \leq V \text{ if } U \supseteq V.$$

We construct a net of points of $A$ by choosing for every $U \in N_x$ a point $x_U \in U \cap A$. It is easy to check that the net $\langle x_U \rangle_{U \in N_x}$, constructed in this way, has $x$ as a limit. \hfill \Box

The proofs of the following theorems are left as exercises.

Theorem 2.9. Let $X$ and $Y$ be topological spaces. A map $f : X \longrightarrow Y$ is continuous if and only if, for all net $\langle x_\alpha \rangle_{\alpha \in I}$ in $X$:

$$f(\lim_{\alpha \in I} x_\alpha) \subseteq \lim_{\alpha \in I} f(x_\alpha).$$

Theorem 2.10. A topological space $X$ is Hausdorff if and only if every net in $X$ has at most one limit.

Nets correspond, for topological spaces, to sequences and, of course, sequences are a particular example of nets in which the directed index set is $\mathbb{N}$. However, when we pass from the definition of subsequence to the definition of subnet we have some differences. For instance, given a sequence there may be subnets which are not subsequences. For this reason, in spite of the fact that by using nets it is possible to extend the concept of convergence to any topological spaces, when dealing with nets we have to be very careful.

Example 2.11. Consider the sequence $\langle (-1)^n \rangle_{n \in \mathbb{N}}$ in the space $\mathbb{R}$ with its standard topology. Clearly, this sequence has 1 and $-1$ as cluster points. We use the construction given in theorem 2.7 to construct a finer net converging to 1. We take as index set:

$$K = \{(n, U) : n \in \mathbb{N} \text{ and } U \text{ is a neighbourhood of 1 and } x_n \in U\} =$$

$$= \{(n, U) : n \text{ even and } U \text{ is a neighbourhood of 1}\},$$

directed by the order:

$$(n, U) \leq (m, V) \text{ if } n \leq m \text{ and } U \supseteq V,$$

and construct a net by defining, for $(n, U)$ in $K$, $x_{(n,U)} = (-1)^n = 1$. This is a subnet of the net we began with and it converges to 1, but, since $|\Sigma| > \omega$, this is not a subsequence. Of course $\langle (-1)^{2m} \rangle_{m \in \mathbb{N}}$ is a subsequence converging to 1.

We conclude this short note with a theorem concerning convergence in product spaces. (This theorem will allow for a fast proof of Tychonoff Theorem.)

Theorem 2.12. A net $\langle x_\alpha \rangle$ in a product $\prod_{\gamma \in I} X_\gamma$ converges to $x$ in the product topology if and only if, for all $\alpha \in I$, the net $\langle \pi_\alpha(x_\gamma) \rangle$ converges to $\pi_\alpha(x)$ in $X_\alpha$. 

Proof. If \( x_\gamma \to x \) in \( \prod_{\gamma \in I} X_\gamma \), then, since \( \pi_\alpha \) is continuous, \( \pi_\alpha(x_\gamma) \to \pi_\alpha(x) \). In order to show the converse, suppose that, for all \( \alpha \in I \), \( \pi_\alpha(x_\gamma) \to \pi_\alpha(x) \) and let

\[
\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})
\]

be a basic neighbourhood of \( x \). Then, for all \( i \in \{1, 2, \ldots, n\} \), there exists \( \gamma_i \) such that, when \( \gamma \geq \gamma_i \), then \( \pi_{\alpha_i}(x_\gamma) \in U_{\alpha_i} \). Hence, if \( \gamma_0 \) is greater than all the \( \gamma_i \)’s (such a \( \gamma \) exists because the index set of the net is directed) we have that, whenever \( \gamma \geq \gamma_0 \), for all \( \alpha \in I \), \( \pi_{\alpha_i}(x_\gamma) \in U_{\alpha_i} \). This means that, when \( \gamma \geq \gamma_0 \), \( x_\gamma \in \bigcap \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \), i.e. \( x_\gamma \to x \). □

References